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Almost Koszul duality and rational conformal field theory

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Almost Koszul Duality and Rational Conformal Field Theory

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Contents

Summary

The core result of this thesis is an explanation of *why* the Dynkin preprojective algebras are almost Koszul. It is shown that both Koszul and almost Koszul preprojective algebras are the functorial image of a universal algebra in a certain monoidal category. In the Dynkin case, this functor factorises, truncating the exact sequences defining Koszulity not in a brutal way, but gently and almost perfectly.

The analysis performed establishes a connection to the $c < 1$ (unitary) boundary conformal field theories, via Pasquier's lattice models, showing that rationality of the theory is intimately connected to almost Koszulity and explaining the presence of the dimensions of the preprojective algebra in the partition function for the theory on a cylinder.

Finally, the analysis is extended, showing that the graphs associated to the $c < 2$ rational boundary conformal field theories should possess almost Koszul algebras. Few examples of almost Koszul algebras are known and one notable achievement of this work is to swell their number.

Preface

I hope, first and foremost, that this thesis talks to mathematicians, particularly representation theorists, explaining some of the properties of the Dynkin preprojective algebras and their $c < 2$ analogues in an interesting diagrammatic way and from a functorial point of view. I hope also that it highlights the important work currently underway in the conformal field theory community and elicits some of the connections to interesting homological questions in the study of quiver algebras.

Secondly, I hope this thesis has something of relevance to say to conformal field theorists. This thesis treads many of the well-worn paths explored by conformal field theory in the past decade in an attempt to understand a homological property of the preprojective algebras. It should certainly not be read as an attempt to tell conformal field theorists what they already know, or even to understand their results from a mathematical point of view. The functorial point of view taken in this thesis is one that is now emerging in published form on the conformal field theory side (although no doubt it has been known for a good deal of time). Largely, this is due to the efforts and understanding of Robert Coquereaux, who makes explicit his aim of building a bridge between several mathematics and physics communities interested in related topics. This thesis will serve, I hope, as a reciprocation from one of those communities and an invitation to a closer dialogue.

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I would have struggled to get this far without the love and unwavering support of my parents and brothers and thanks also to my son Kirin for his unconditional love and who is a constant reminder of the joys of life and the need for the appropriate work-life balance. Finally and most importantly, thanks and all my love to my wife Rachel, whose enduring love, support and understanding have been my strength throughout our ten years together.

Chapter 1

Introduction

1.1 The main results

The primary objects of study in this thesis are functors on two (families of) $\mathbb{C}$-linear monoidal categories $\underline{\mathrm{TL}}$ and $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. Morphisms in both categories are described by equivalence classes of diagrams, whose composition laws depend on a parameter $q \in \mathbb{C} \setminus \{ 0 \}$. The Temperley-Lieb category $\underline{\mathrm{TL}}$ is well-known and is equivalent to the tensor subcategory of $\mathcal{U}_q[\mathfrak{sl}_2]$-$\underline{\mathrm{mod}}$ generated by the fundamental representation of $\mathcal{U}_q[\mathfrak{sl}_2]$. The category $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ is an $\mathfrak{sl}_3$ analogue of $\underline{\mathrm{TL}}$. Here a few words are perhaps in order, to better explain the relationship between the diagram categories $\underline{\mathrm{TL}}$ and $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ and their quantum group counterparts $\mathcal{U}_q[\mathfrak{sl}_2]$-$\underline{\mathrm{mod}}$ and $\mathcal{U}_q[\mathfrak{sl}_3]$-$\underline{\mathrm{mod}}$.

Morphisms in the Temperley-Lieb category are equivalence classes of $\mathbb{C}$-linear combinations of planar Brauer diagrams, essentially describing the different ways of pairing dots in such a way that a planarity condition is satisfied. Constructing a $\mathbb{C}$-linear monoidal functor on $\underline{\mathrm{TL}}$ is greatly facilitated by knowing a set of generators and relations for the category. For the Temperley-Lieb category this is straightforward, however in moving to $\mathfrak{sl}_3$ some problems are encountered.

The first issue is finding the correct analogue of the Temperley-Lieb category, that is, a category that is equivalent to the tensor subcategory of $\mathcal{U}_q[\mathfrak{sl}_3]$-$\underline{\mathrm{mod}}$ generated by the two fundamental representations of $\mathcal{U}_q[\mathfrak{sl}_3]$. A candidate appears in the literature, namely the categorical version of Kuperberg's A_2 spider. Secondly, there is the question of generators and relations for this category. This thesis does not attempt to address these issues, rather, it bypasses them by presenting a diagram category, $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$, explicitly in terms of generators and relations.

Neither $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ nor $\underline{\mathrm{TL}}$ are additive categories, however by suitably completing them (which includes adjoining the images of idempotents), sensible questions about their semisimplicity can be posed. In this thesis, the completion is achieved explicitly by using the Yoneda functor to embed the original diagram category, $\mathcal{C} = \underline{\mathrm{TL}}$ or $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$, into its 'module category', the category of contravariant functors $\mathfrak{Fun}^\circ(\mathcal{C}, \underline{\mathrm{Vect}})$.

The parameter $q \in \mathbb{C} \setminus \{0\}$, on which the categories $\underline{\mathrm{TL}}$ and $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ depend, plays a critical role in the analysis: q is said to be *singular* if $q \neq \pm 1$ and q is a root of unity; q is said to be *generic* otherwise. For singular q there is a smallest positive integer h such that $q^{2h} = 1$.

The following results, well known for $\underline{\mathrm{TL}}$, are proved:

Theorem 1.1.1. *For generic values of q, there is a sequence of simple functors $\{X_i\}_{i=0}^{\infty}$ such that every object $\underline{n}$ in $\underline{TL}$ has a canonical decomposition in $\mathfrak{Fun}^\circ(\underline{TL}, \underline{Vect})$:*

$$\underline{n} \cong \bigoplus_{i \geq 0} X_i \otimes_{\mathbb{C}} \mathrm{Hom}(X_i, \underline{n}) \ .$$

Theorem 1.1.2. *For generic values of q, there is a collection of simple functors $\{X_{i,j}\}_{i,j \geq 0}$ such that every object A in $\underline{Fus}_{\mathfrak{sl}_3}$ has a canonical decomposition in $\mathfrak{Fun}^\circ(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$:*

$$A \cong \bigoplus_{i,j \geq 0} X_{i,j} \otimes_{\mathbb{C}} \mathrm{Hom}(X_{i,j}, A) \ .$$

Intimately connected to the proofs of the above results are the following:

Theorem 1.1.3. *The monoidal structure on $\underline{TL}$ induces a monoidal structure on the X_i, for which the X_i satisfy the $\mathfrak{sl}_2$ fusion rule: $X_i \otimes X_1 = X_{i+1} \oplus X_{i-1}$.*

Theorem 1.1.4. *The monoidal structure on $\underline{Fus}_{\mathfrak{sl}_3}$ induces a monoidal structure on the $X_{i,j}$, for which the $X_{i,j}$ satisfy the $\mathfrak{sl}_3$ fusion rule:*

$$\begin{aligned} X_{i,j} \otimes X_{1,0} &= X_{i+1,j} \oplus X_{i-1,j+1} \oplus X_{i,j-1} \ , \\ X_{i,j} \otimes X_{0,1} &= X_{i,j+1} \oplus X_{i+1,j-1} \oplus X_{i-1,j} \ . \end{aligned}$$

Here, the convention is taken that if an index is negative then the corresponding object is the zero object.

Theorem 1.1.4 gives an *a posteriori* reason for the name $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$.

In the $\mathfrak{sl}_2$ case, there are maps $X_i \otimes X_j \to X_{i+j}$, projecting to the direct summand of maximal degree, that define a graded associative product on $X = \bigoplus_{i\geq 0} X_i$. This graded algebra possesses an interesting property (see [MOV06, Man87]):

Theorem 1.1.5. *The graded algebra* $X = \bigoplus_{i\geq 0} X_i$ *is Koszul.*

In the language of non-commutative geometry, X corresponds to the coordinate ring of the quantum plane. For $\mathfrak{sl}_3$, the coordinate rings of quantum 3-space and its dual have analogues in $\bigoplus_{i\geq 0} X_{i,0}$ and $\bigoplus_{j\geq 0} X_{0,j}$, possessing the same property (cf. [Man87]):

Theorem 1.1.6. *The graded algebras* $\bigoplus_{i\geq 0} X_{i,0}$ *and* $\bigoplus_{j\geq 0} X_{0,j}$ *are Koszul.*

In the $\underline{\mathrm{TL}}$ case, the algebra X is the 'universal preprojective algebra' in the following sense (see also [MOV06]):

Corollary 1.1.7. *The preprojective algebra* Π *of a non-Dynkin symplectic quiver* Q *is the 'image' of* X *under a* $\mathbb{C}$*-linear monoidal functor* $\underline{TL} \to S\text{-}\underline{mod}\text{-}S$*, where* S *is the algebra of paths of length zero in* Q*. Consequently,* Π *is Koszul.*

As with semisimplicity, 'image' needs to be interpreted in the correct way: X is not an object in $\underline{\mathrm{TL}}$, rather it is in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$, but the monoidal functor can be extended to X.

Of considerably more interest is the case when q is singular. The (completed) categories $\underline{\mathrm{TL}}$ and $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ are no longer semisimple, however it is possible to identify a proper tensor ideal, the ideal of negligible morphisms, for which the quotient category (appropriately completed) is semisimple:

Theorem 1.1.8. *Let* $\underline{TL}_{\mathrm{red}}$ *denote the quotient of* $\underline{TL}$ *by the ideal of negligible morphisms. There is a finite sequence of simple functors,* $\{R(X_i)\}_{0\leq i\leq h-2}$*, such that every object* $\underline{n}$ *in* $\underline{TL}_{\mathrm{red}}$ *has a canonical decomposition in* $\mathfrak{Fun}^\circ(\underline{TL}_{\mathrm{red}}, \underline{Vect})$*:*

$$\underline{n} \cong \bigoplus_{i=0}^{h-2} R(X_i) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{red}}(R(X_i), \underline{n}) \ ,$$

where $\mathrm{Hom}_{\mathrm{red}}$ *denotes morphisms in* $\mathfrak{Fun}^\circ(\underline{TL}_{\mathrm{red}}, \underline{Vect})$*. The functors* $R(X_i)$ *satisfy a truncated* $\mathfrak{sl}_2$ *fusion rule, that is, the* $\mathfrak{sl}_2$ *fusion rule with the additional constraint that* $X_{h-1} = 0$.

Identifying $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}_{\mathrm{red}}, \underline{\mathrm{Vect}})$ with the subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ whose objects vanish on the ideal of negligible morphisms, $R(X_i)$ is identified with the 'top' of the functor X_i. An analogous result holds for $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$:

Theorem 1.1.9. *Let $\underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3}$ denote the quotient of $\underline{Fus}_{\mathfrak{sl}_3}$ by the ideal of negligible morphisms. There is a finite collection of simple functors, $\{R(X_{i,j})\}_{0\leq i,j\leq h-3}$, such that every object A in $\underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3}$ has a canonical decomposition in $\mathfrak{Fun}^{\circ}(\underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3}, \underline{Vect})$:*

$$A \cong \bigoplus_{0\leq i,j\leq h-3} R(X_{i,j}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{red}}(R(X_{i,j}), A) ,$$

where $\mathrm{Hom}_{\mathrm{red}}$ *denotes morphisms in $\mathfrak{Fun}^{\circ}(\underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3}, \underline{Vect})$. The functors $R(X_{i,j})$ satisfy a truncated $\mathfrak{sl}_3$ fusion rule, that is, the $\mathfrak{sl}_3$ fusion rule with the additional constraint that $X_{i,j} = 0$ whenever $i + j = h - 2$.*

The main results of this thesis assert that Koszulity is almost preserved in the quotient:

Theorem 1.1.10. *The graded algebra $R(X) = \bigoplus_{0\leq i\leq h-2} R(X_i)$ in $\mathfrak{Fun}^{\circ}(\underline{TL}, \underline{Vect})$ is almost Koszul.*

Theorem 1.1.11. *The graded algebras $\bigoplus_{0\leq i\leq h-3} R(X_{i,0})$ and $\bigoplus_{0\leq j\leq h-3} R(X_{0,j})$ in $\mathfrak{Fun}^{\circ}(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$ are almost Koszul.*

Here, 'almost Koszul' means something precise ([BBK02]): an algebra X is almost Koszul if there exists a coalgebra A such that the bigraded differential complex $(X_p \otimes A_q)_{p,q\geq 0}$, with the Koszul differential δ, is bounded (at degrees m and n say) and is exact except in degrees $(0, 0)$ and (m, n).

For $\underline{\mathrm{TL}}$, this furnishes a new proof that the Dynkin preprojective algebras are almost Koszul and moreover, highlights precisely what is happening in these singular cases:

Corollary 1.1.12. *Let Q be a Dynkin quiver together with a symplectic form ω. The preprojective algebra Π of (Q, ω) is the 'image' of X under a $\mathbb{C}$-linear monoidal functor $\underline{TL} \to S\text{-}\underline{mod}\text{-}S$ factoring through the quotient category $\underline{TL}_{\mathrm{red}}$. Consequently, Π is almost Koszul.*

It should be emphasized that, in the case of the Dynkin quivers, almost Koszulity for the preprojective algebras is not a case of something going wrong, rather something very particular goes *right.*

The Dynkin preprojective algebras nearly exhaust the known examples of almost Koszul algebras. However, the analysis of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ for singular values of q permits the construction of many more examples. What is required is to identify appropriate quivers Q and build (certain) $\mathbb{C}$-linear monoidal functors $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3} \to S\text{-}\underline{\mathrm{mod}}\text{-}S$ that factor through the quotient category. Sufficient conditions for the existence of such a functor are detailed in Appendix B.

Remarkably, the classification of $1 < c < 2$ rational conformal field theories has produced a collection of quivers that appear to be suitable candidates ([Zub02]) and Ocneanu ([Ocn00]) claims to have shown that these quivers satisfy the sufficient conditions, as part of his theory of flat connections on cell complexes. Thus, there is a large number of new almost Koszul algebras awaiting further examination.

The relation between almost Koszulity and rational conformal field theory is further highlighted by results detailing how these almost Koszul algebras govern the behaviour of the partition function for lattice models of such theories. The theorem is suggested by a result first brought to the author's attention by Patrick Dorey ([Dor93]):

Theorem 1.1.13. *Let (Q, ω) be a non-Dynkin symplectic quiver and define Π to be the corresponding preprojective algebra. Then the n^{th} graded piece of the path algebra of Q is semisimple as a representation of the Temperley-Lieb algebra $TL_n = \mathrm{End}(\underline{n})$ and decomposes according to*

$$(\mathbb{C}Q)_n \cong \bigoplus_{p=0}^{\infty} \Pi_p \otimes_{\mathbb{C}} \mathrm{Hom}(X_p, \underline{n}) \ .$$

For Q a Dynkin quiver with Coxeter number h and Π the preprojective algebra, the graded pieces of the path algebra of Q decompose as a finite *sum of simple Temperley-Lieb representations,*

$$(\mathbb{C}Q)_n \cong \bigoplus_{p=0}^{h-2} \Pi_p \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{red}}(R(X_p), \underline{n}) \ ,$$

where Hom_{red} denotes morphisms in the quotient (functor) category.

This theorem is obtained from Theorems 1.1.1 and 1.1.8 by applying the $\mathbb{C}$-linear monoidal functor of Corollaries 1.1.7 and 1.1.12. The remaining statement is that $\mathrm{Hom}(X_p, \underline{n})$ and $\mathrm{Hom}(R(X_p), \underline{n})$ are simple $\mathrm{End}(\underline{n})$-modules.

The transfer matrix of the lattice model is an element of the Temperley-Lieb algebra, hence the above formulae detail its decomposition (and hence that of the partition function) on the invariant spaces of the theory. Analogous formulae hold for the partition function of the limiting boundary conformal field theory, where this exists. The Dynkin quivers yield examples of *rational* lattice models: the decomposition in Theorem 1.1.13 remains finite as n tends to infinity. Both rationality and that the algebra-coalgebra pair (X, A) is almost Koszul rather than Koszul, follow immediately from the fact that the $\mathbb{C}$-linear monoidal functor defining the lattice model factors through the ideal of negligible morphisms.

For the $\mathfrak{sl}_3$ case, the corresponding statements upon applying a $\mathbb{C}$-linear monoidal functor $\underline{\text{Fus}}_{\mathfrak{sl}_3} \to S\text{-}\underline{\text{mod}}\text{-}S$ to an object A in $\underline{\text{Fus}}_{\mathfrak{sl}_3}$ and using Theorems 1.1.2 and 1.1.9, detail the decomposition of paths on the doubled quiver $\overline{Q}$ (or both forward and backward-moving paths on the original quiver Q) into simple representations of $\text{End}(A)$. As in the Temperley-Lieb case, these functors define lattice models, which are rational provided the functor factors through the ideal of negligible morphisms.

1.2 Structure of the thesis

This thesis charts the tale of how, in this story at least, Koszul and almost Koszul algebras are closer bedfellows than might be imagined. In particular, I hope that the enduring impression will be that, rather than being defective, the almost Koszul algebras described arise in a very special way. The progression throughout is largely from the familiar to the less so and thus, the Temperley-Lieb case, in which much of what is happening is already understood (at least piecemeal, if not as a whole), is developed in detail before progressing to the category $\underline{\text{Fus}}_{\mathfrak{sl}_3}$. The treatment of the material presented in this thesis is very much my own personal take on it and I have, where possible, tried to provide references to relevant work in the literature. My sincerest apologies to those whose contributions I have overlooked.

Chapter 2 introduces the notion of a Koszul algebra-coalgebra pair and elicits a recurrence relation on the graded pieces of the algebra and coalgebra that is a necessary condition for them to form a Koszul pair. A particular form of this recurrence is analysed in Chapter 3, whose solutions fall into a finite-tame-wild trichotomy. A number of combinatorial results are proved here that are crucial to the successful pursuit of the theorems detailed previously in this Introduction. Chapter 4 identifies almost Koszul algebra-coalgebra pairs as the possible source of finite solutions to the recurrences of Chapter 3, as well as proving that there is an almost Koszul pair associated with the tadpole quivers. The core of the thesis is Chapter 5, where the Temperley-Lieb case is studied in detail and the necessary formalism for interpreting Theorems 1.1.1-1.1.13 is developed. Chapter 6 is a digression to discuss the connections with lattice models for rational conformal field theories, before returning to the analysis of the category $\underline{\text{Fus}}_{\mathfrak{sl}_3}$ in Chapter 7. Appendices A-D present a number of important results whose inclusion in the main text would have interrupted the natural flow of the thesis.

1.3 Quiver preliminaries

In this section some of the basic material about quivers is presented. A QUIVER Q is a pair of sets Q_0 and Q_1 together with maps

$$Q_1 \overset{h}{\underset{t}{\rightrightarrows}} Q_0 \; .$$

Quivers are often presented pictorially with a vertex for every element of Q_0 and an arrow for every element $a \in Q_1$ with tail at vertex $t(a)$ and head at vertex $h(a)$. The notation a_{ij} will be used to denote an arrow a with $t(a) = i$ and $h(a) = j$.

The quiver Q is said to be FINITE if Q_0 and Q_1 are finite sets. Let $Q_0 = A_0 \sqcup B_0$ and $Q_1 = A_1 \sqcup B_1$ be partitions of Q_0 and Q_1 satisfying

$$\begin{aligned} h(A_1) \subset A_0 \,, \qquad h(B_1) \subset B_0 \quad , \\ t(A_1) \subset A_0 \,, \qquad t(B_1) \subset B_0 \quad . \end{aligned} \tag{1.1}$$

The quiver Q is CONNECTED if the only partitions of Q_0 and Q_1 satisfying the inclusions (1.1) are trivial, that is, one of the sets A_0 or B_0 is empty. A quiver that is not connected is said to be DISCONNECTED. A quiver Q defines two important algebras over $\mathbb{C}$. The semisimple algebra $S = S(Q_0)$ is defined by

$$S = \bigoplus_{i \in Q_0} \mathbb{C}_{(i)} \; .$$

Denote by e_i the unit $1 \in \mathbb{C}_{(i)}$. By construction e_i is an idempotent. If Q is finite then S is a unital algebra with unit

$$1 = 1_S = \sum_{i \in Q_0} e_i \; .$$

The set Q_1 defines a bimodule V for S with $\mathbb{C}$-basis $\{v_a \ : \ a \in Q_1\}$. A bimodule action of S on V is given by

$$e_i \cdot v_a \cdot e_j = \begin{cases} v_a & \text{if } i = t(a) \text{ and } j = h(a) \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

A common abuse of notation will be to write $a \in Q_1$ for the basis element $v_a \in V$. The path algebra of Q, denoted $\mathbb{C}Q$, is defined to be the tensor algebra of V over S,

$$\mathbb{C}Q = T(V) = S \oplus V \oplus (V \otimes_S V) \oplus (V \otimes_S V \otimes_S V) \oplus \cdots .$$

The algebra $\mathbb{C}Q$ is non-commutative and has the obvious grading. Denote the graded piece with degree n by $(\mathbb{C}Q)_n$. An element of $\mathbb{C}Q$ of the form

$$v_{a_1} \cdots v_{a_n} \ , \qquad a_1, \ldots, a_n \in Q_1 \ ,$$

is called a PATH of length n and corresponds to the combinatorial notion of a path on the quiver. The collection of paths of arbitrary length form a $\mathbb{C}$-basis for the path algebra $\mathbb{C}Q$. A quiver Q is said to be PATH-CONNECTED (or STRONGLY CONNECTED) if for every pair of vertices i and j, the vector subspace $e_i(\mathbb{C}Q)e_j \subset \mathbb{C}Q$ is not the zero subspace.

A symplectic quiver is a quiver Q together with a map $\omega \colon Q_1 \times Q_1 \to \mathbb{C}$ satisfying

1. $\omega(a_{ij}, b_{kl}) = -\omega(b_{kl}, a_{ij})$ for all arrows a_{ij} and b_{kl};
2. for every arrow a_{ij} there exists a unique arrow b_{ji} such that $\omega(a_{ij}, b_{ji}) \neq 0$.

The map ω determines an S-bilinear non-degenerate symplectic form

$$\begin{aligned} V \times V &\longrightarrow S \\ (a_{ij}, b_{kl}) &\longmapsto e_i \omega(a_{ij}, b_{kl}) e_l \ . \end{aligned}$$

As a consequence of condition 2, this symplectic form factorises through an S-bilinear map $V \otimes_S V \to S$. The notion of a symmetric quiver is entirely analogous, with a plus sign instead of a minus sign in condition 1.

Given a symplectic (symmetric) form ω for the quiver Q, let $\overline{a}_{ji}$ denote the element of the dual basis for V defined uniquely by

$$\omega(\overline{a}_{ji}, b_{ij}) = \begin{cases} 1 & \text{if } b_{ij} = a_{ij} \ , \\ 0 & \text{otherwise.} \end{cases}$$

By condition 2 the element $\overline{a}_{ji}$ is a non-zero multiple of an arrow from j to i.

The preprojective algebra of a symplectic quiver (Q, ω) is the quotient of the path algebra by the ideal of relations of the form

$$\sum_{c_{pi}} \overline{c}_{ip} c_{pi} \,, \qquad i \in Q_0 \,,$$

where the sum is taken over all arrows c with head at the fixed vertex i and tail at another vertex p. It is easy to show that different choices for ω always produce isomorphic preprojective algebras.

The adjacency matrix of a quiver Q is the matrix A with entries

$$A_{ij} = \dim e_i (\mathbb{C}Q)_1 e_j \ .$$

A quiver Q is path-connected if and only if its adjacency matrix is irreducible ([DM67]).

An irreducible non-negative real matrix A possesses an eigenvalue equal to its spectral radius $\rho(A)$. This eigenvalue is called the Perron-Frobenius eigenvalue and an associated eigenvector is called a Perron-Frobenius vector. A Perron-Frobenius eigenvector is nowhere zero and all its components are real and have the same sign ([BP79]).

Henceforth, all quivers will be assumed to be finite and path-connected.

Chapter 2

Koszul duality

The story begins with Koszul duality and the classical example of the duality between $\mathrm{Sym}(V)$ and $\Lambda(V)$. This is the foundation on which more complicated examples can be built, namely Koszul duality for the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. The chapter concludes with the introduction of a matrix recurrence, a consequence of Koszul duality and analysis of which plays an important role in the thesis.

2.1 Classical Koszul duality

2.1.1 Koszul pairs

The most important concept in this thesis is that of Koszul algebra-coalgebra pairs (and later that of almost Koszul pairs). Let S be a finite-dimensional semisimple algebra over $\mathbb{C}$. Fix a graded algebra $(\Pi_\bullet, \mu)$ over S and a graded coalgebra $(\Lambda_\bullet, \Delta)$ over S (it will always be assumed that $\Pi_0 = S = \Lambda_0$) and suppose that $\Pi_1 \cong \Lambda_1$. Consider the bigraded complex $\Pi_\bullet \otimes_S \Lambda_\bullet$ equipped with the maps $\delta_{p,q}$ defined by the following commutative diagram:

$$\begin{tikzcd}
\Pi_p \otimes_S \Lambda_q \arrow[rrr, "\delta_{p,q}"] \arrow[dr, "1 \otimes \Delta_{1,q-1}"] & & & \Pi_{p+1} \otimes_S \Lambda_{q-1} \\
& \Pi_p \otimes_S \Lambda_1 \otimes_S \Lambda_{q-1} \arrow[r, "\cong"] & \Pi_p \otimes_S \Pi_1 \otimes_S \Lambda_{q-1} \arrow[ur, "\mu_{p,1} \otimes 1"] &
\end{tikzcd}$$

If δ is a differential then $\Pi_\bullet \otimes_S \Lambda_\bullet$ is called the KOSZUL COMPLEX and δ will be called the KOSZUL DIFFERENTIAL ([Man87]). The pair (Π, Λ) is said to be KOSZUL if $\Pi_\bullet \otimes_S \Lambda_\bullet$ is an *exact* differential complex, except at degree $(0, 0)$.

Remark 2.1.1. *It is more usual to define Koszulity for a pair of algebras, where one of them is dualised (in an appropriate sense) to obtain the coalgebra (cf. [BGS96] and the original definition of a Koszul algebra due to Priddy [Pri70]).*

Remark 2.1.2. *It is easily shown that the Koszul map δ is a differential if (and only if) the composite*

$$\Lambda_2 \longrightarrow \Pi_1 \otimes_S \Lambda_1 \longrightarrow \Pi_2$$

is zero.

Occasionally, an algebra Π will be described as Koszul. Thus, the statement is that there exists a coalgebra Λ such that (Π, Λ) is a Koszul algebra-coalgebra pair. In fact, the coalgebra Λ is unique up to isomorphism and is the quadratic dual coalgebra to Π.

2.1.2 The classical example

The primary example of a Koszul algebra-coalgebra pair is that of the symmetric tensor algebra and alternating tensor coalgebra of a finite-dimensional $\mathbb{C}$-vector space. (This is more usually stated as the symmetric algebra $\mathrm{Sym}^\bullet(V)$ and the exterior algebra $\Lambda^\bullet(V^*)$ are Koszul dual algebras (see Remark 2.1.1) and is well-known [BGS96, PP05].)

Let V be a finite-dimensional $\mathbb{C}$-vector space and consider the symmetric tensor algebra $\mathrm{Sym}^\bullet(V)$ and the alternating tensor coalgebra $\Lambda^\bullet(V)$. It is shown that the algebra-coalgebra pair $(\mathrm{Sym}^\bullet(V), \Lambda^\bullet(V))$ is Koszul. Denote the product on $\mathrm{Sym}^\bullet(V)$ by μ and the coproduct on $\Lambda^\bullet(V)$ by Δ.

Explicitly, fix a basis $v_1, \dots, v_n$ for V. Then monomials in the v_i form a basis for $\mathrm{Sym}^\bullet(V)$. Let I be an ordered set $\{i_1 < \cdots < i_q\} \subset \{1, \dots, n\}$. Then elements of the form

$$v^I \stackrel{\text{def}}{=} v_{i_1} \wedge \cdots \wedge v_{i_q}$$

are a basis for $\Lambda^q(V)$. The coproduct Δ is defined by

$$\begin{aligned} \Delta \colon \Lambda_{p+q} &\longrightarrow \Lambda_p \otimes \Lambda_q \\ v^I &\longmapsto \sum_{K \subset I} \mathrm{sgn}_I(K) v^K \otimes v^{I \setminus K} \,, \end{aligned}$$

where the sum runs over all ordered subsets K of I of cardinality p and the function sgn_I is defined by

$$\mathrm{sgn}_I(K) = \prod_{k \in K} (-1)^{\#\{i \in I \setminus K \,:\, i < k\}} \,.$$

Thus, $\mathrm{sgn}_I(K)$ keeps track of the number of times the anticommutation relations are used to reorder v^I as $v^K \otimes v^{I\setminus K}$. By convention, $\mathrm{sgn}_I(\emptyset) = 1$.

It is straightforward to show that Δ is coassociative.

Consider the bigraded space $\mathrm{Sym}^\bullet(V)\otimes_{\mathbb{C}}\Lambda^\bullet(V)$ and the collection of morphisms defined by

$$\delta_{p,q}\colon\ \mathrm{Sym}^p(V)\otimes_{\mathbb{C}}\Lambda^q(V) \xrightarrow{1\otimes\Delta} \mathrm{Sym}^p(V)\otimes_{\mathbb{C}} V\otimes_{\mathbb{C}}\Lambda^{q-1}(V) \xrightarrow{\mu\otimes 1} \mathrm{Sym}^{p+1}(V)\otimes_{\mathbb{C}}\Lambda^{q-1}(V)$$

These make $\mathrm{Sym}^\bullet(V) \otimes_{\mathbb{C}} \Lambda^\bullet(V)$ into a bigraded differential complex. This is easily shown in degree 2, which suffices to prove the result in all degrees (see Remark 2.1.2).

To show that the Koszul differential is exact, notice first that it operates on $v_{j_1}v_{j_2}\cdots v_{j_p}\otimes v_{1_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_q}$ by moving vectors from one side of the tensor to the other and hence preserves the union of the indexing sets. Fix therefore a multi-index K and define

$$\underline{v}^I \stackrel{\text{def}}{=} v_{j_1}\cdots v_{j_p}\otimes v_{i_1}\wedge\cdots\wedge v_{i_q} \in \mathrm{Sym}^p(V)\otimes\Lambda^q(V)\ ,$$

where $I = \{i_1 < i_2 < \cdots < i_q\} \subset K$ and $K\setminus I = \{j_1 \le j_2 \le \cdots \le j_p\}$. Then

$$\delta(\underline{v}^I) = \sum_{i\in I} \mathrm{sgn}_I(\{i\})\, \underline{v}^{I\setminus\{i\}}\ .$$

Let $w = \sum_I a_I \underline{v}^I \in \mathrm{Sym}^p(V)\otimes\Lambda^q(V)$, where the index runs over all ordered subsets I of K with cardinality q and where $a_I \in \mathbb{C}$. Suppose that $w \in \mathrm{Ker}\,\delta$. Then

$$\begin{aligned}
0 &= \delta(w)\\
&= \sum_I a_I \sum_{i\in I} \mathrm{sgn}_I(\{i\})\underline{v}^{I\setminus\{i\}}\\
&= \sum_{\substack{I\\ 1\in I}} a_I \sum_{i\in I} \mathrm{sgn}_I(\{i\})\underline{v}^{I\setminus\{i\}} + \sum_{\substack{I\\ 1\notin I}} a_I \sum_{i\in I} \mathrm{sgn}_I(\{i\})\underline{v}^{I\setminus\{i\}}\ .
\end{aligned}$$

Wedging with v_1 on the right now yields

$$\begin{aligned}
0 &= \sum_{\substack{I\\ 1\in I}} a_I \sum_{i\in I} \mathrm{sgn}_I(\{i\})\underline{v}^{I\setminus\{i\}}\wedge v_1 + \sum_{\substack{I\\ 1\notin I}} a_I \sum_{i\in I} \mathrm{sgn}_I(\{i\})\underline{v}^{I\setminus\{i\}}\wedge v_1\\
&= \sum_{\substack{I\\ 1\in I}} a_I \underline{v}^{I\setminus\{1\}}\wedge v_1 + \sum_{\substack{I\\ 1\notin I}} a_I \sum_{i\in I} \mathrm{sgn}_I(\{i\})\underline{v}^{I\setminus\{i\}}\wedge v_1\ .
\end{aligned}$$

Reorder the tensor (anticommuting v_1 past $q-1$ vectors) and rearrange to obtain

$$\sum_{\substack{I\\1\in I}} a_I \underline{v}^I = -\sum_{\substack{I\\1\notin I}} a_I \sum_{i\in I} \operatorname{sgn}_I(\{i\}) \underline{v}^{(I\cup\{1\})\setminus\{i\}} . \tag{2.1}$$

To show that $w \in \operatorname{Im}\delta$, consider

$$u = \sum_{\substack{I\\1\notin I}} a_I \underline{v}^{I\cup\{1\}} .$$

Then $\delta(u) = w$. Explicitly,

$$\begin{aligned}
\delta(u) &= \sum_{\substack{I\\1\notin I}} a_I \delta(\underline{v}^{I\cup\{1\}}) \\
&= \sum_{\substack{I\\1\notin I}} a_I \underline{v}^I - \sum_{\substack{I\\1\notin I}} a_I \sum_{i\in I} \operatorname{sgn}_I(\{i\}) \underline{v}^{(I\cup\{1\})\setminus\{i\}} \\
&= \sum_{\substack{I\\1\notin I}} a_I \underline{v}^I + \sum_{\substack{I\\1\in I}} a_I \underline{v}^I \qquad (2.2) \\
&= w ,
\end{aligned}$$

where (2.2) is obtained using (2.1).

Thus $(\mathrm{Sym}^\bullet(V), \Lambda^\bullet(V))$ is indeed a Koszul pair.

2.2 More examples of Koszul pairs

It is possible to use the classical case to construct more complicated examples of Koszul pairs. The following is my own analysis of this process (see also [MV01]). The section concludes with the first sighting of the affine Dynkin diagrams.

2.2.1 Twisted bimodules

Let A be a bialgebra over $\mathbb{C}$. For a left A-module V, define the twisted A-bimodule, $V\widetilde{\otimes}A$ as follows:

$$\begin{array}{rl}
\text{as a vector space} & \mathrm{V}\,\widetilde{\otimes} A = V \otimes_{\mathbb{C}} A , \\
\text{as an } A\text{-bimodule} & \alpha \cdot (v \otimes a) \cdot \beta = \alpha_{(1)} v \otimes \alpha_{(2)} a \beta ,
\end{array}$$

where Sweedler's notation for the coproduct is used, together with the summation convention,

$$\Delta(\alpha) = \alpha_{(1)} \otimes \alpha_{(2)} = \sum \alpha_{(1)} \otimes \alpha_{(2)} \in A \otimes A\,.$$

Proposition 2.2.1. *Let V and W be left A-modules. Then $V \otimes W$ is a left A-module and there is an isomorphism of A-bimodules*

$$\begin{array}{rcl} \phi\colon (V\tilde{\otimes}A) \otimes_A (W\tilde{\otimes}A) & \longrightarrow & (V \otimes W)\tilde{\otimes}A \\ v \otimes a \otimes w \otimes b & \longmapsto & v \otimes a_{(1)}w \otimes a_{(2)}b\,. \end{array}$$

Proof. First it is shown that ϕ is A-linear. Explicitly

$$\begin{aligned} \phi(\alpha \cdot v \otimes a \otimes w \otimes b \cdot \beta) &= \phi(\alpha_{(1)}v \otimes \alpha_{(2)}a \otimes w \otimes b\beta) \\ &= \alpha_{(1)}v \otimes (\alpha_{(2)}a)_{(1)}w \otimes (\alpha_{(2)}a)_{(2)}b\beta \\ &= \alpha_{(1)}v \otimes \alpha_{(2)(1)}a_{(1)}w \otimes \alpha_{(2)(2)}a_{(2)}b\beta \qquad (2.3) \\ &= \alpha_{(1)(1)}v \otimes \alpha_{(1)(2)}a_{(1)}w \otimes \alpha_{(2)}a_{(2)}b\beta \qquad (2.4) \\ &= \alpha_{(1)} \cdot (v \otimes a_{(1)}w) \otimes \alpha_{(2)}a_{(2)}b\beta \\ &= \alpha \cdot v \otimes a_{(1)}w \otimes a_{(2)}b\beta \\ &= \alpha \cdot \phi(v \otimes a \otimes w \otimes b) \cdot \beta\,. \end{aligned}$$

Notice that (2.3) requires that Δ be an algebra homomorphism and (2.4) uses coassociativity. It now remains to show that ϕ is a bijection. Consider the map

$$\begin{array}{rcl} \tilde{\phi}\colon (V \otimes W)\tilde{\otimes}A & \longrightarrow & (V\tilde{\otimes}A) \otimes_A (W\tilde{\otimes}A) \\ v \otimes w \otimes a & \longmapsto & v \otimes 1 \otimes w \otimes a\,. \end{array}$$

Then $\tilde{\phi}$ is the inverse map to ϕ since

$$\begin{aligned} \tilde{\phi}\phi(v \otimes a \otimes w \otimes b) &= \tilde{\phi}(v \otimes a_{(1)}w \otimes a_{(2)}b) \\ &= v \otimes 1 \otimes a_{(1)}w \otimes a_{(2)}b \\ &= v \otimes a \otimes w \otimes b \end{aligned}$$

and

$$\begin{aligned} \phi\tilde{\phi}(v \otimes w \otimes a) &= \phi(v \otimes 1 \otimes w \otimes a) \\ &= v \otimes w \otimes a\,, \end{aligned}$$

whence the result. □

Let $f\colon V \to W$ be a morphism of left A-modules. Then $f \otimes 1\colon V\widetilde{\otimes}A \to W\widetilde{\otimes}A$ is a morphism of A-bimodules. Explicitly

$$\begin{aligned} f \otimes 1(\alpha \cdot v \otimes a \cdot \beta) &= f\alpha_{(1)}v \otimes \alpha_{(2)}a\beta \\ &= \alpha_{(1)}fv \otimes \alpha_{(2)}a\beta \\ &= \alpha \cdot fv \otimes a \cdot \beta\,. \end{aligned}$$

Hence $\bullet\widetilde{\otimes}A$ defines a functor A-<u>mod</u> $\to$ A-<u>mod</u>-A. Notice that $\bullet\widetilde{\otimes}A$ is exact since exactness only depends on the underlying vector space structure.

2.2.2 Koszul pairs and the twisted bimodule functor

Let Π be a graded algebra over $\mathbb{C}$. Suppose further that Π is a graded left A-module and that the product $\mu\colon \Pi\otimes\Pi \to \Pi$ is a (graded) A-linear map (with Δ determining the action on the tensor product). Then $\Pi\widetilde{\otimes}A$ is a graded algebra over A for the product given by $\widetilde{\mu} = (\mu \otimes 1)\phi$:

$$\widetilde{\mu}((v \otimes a) \otimes (w \otimes b)) = \mu(v \otimes a_{(1)}w) \otimes a_{(2)}b\,.$$

The map $\widetilde{\mu}$ is well-defined and is certainly A-linear (using that μ is A-linear). Moreover, it is easily shown that $\widetilde{\mu}$ is associative (using that μ is A-linear again and coassociativity of Δ).

Analogously, let Λ be a graded coalgebra over $\mathbb{C}$ and suppose further that Λ is a graded left A-module and that the coproduct $\Delta\colon \Lambda \to \Lambda\otimes\Lambda$ is a (graded) A-linear map. Then $\Lambda\widetilde{\otimes}A$ is a graded coalgebra over A for the coproduct given by $\widetilde{\Delta} = \widetilde{\phi}(\Delta \otimes 1)$:

$$\widetilde{\Delta}(v \otimes a) = (v_{(1)} \otimes 1) \otimes (v_{(2)} \otimes a)\,.$$

The map $\widetilde{\Delta}$ is A-linear and coassociative.

Theorem 2.2.2. *Let (Π, Λ) be a Koszul algebra-coalgebra pair and suppose that Π and Λ are graded left A-modules for a $\mathbb{C}$-bialgebra A. If the product on Π and the coproduct on Λ are (graded) A-linear maps then $(\Pi\widetilde{\otimes}A, \Lambda\widetilde{\otimes}A)$ is a Koszul algebra-coalgebra pair.*

Proof. The bigraded differential complex $(\Pi_\bullet \otimes \Lambda_\bullet)\widetilde{\otimes}A$ with differential given by $\delta \otimes 1$ for δ the Koszul differential on $\Pi_\bullet \otimes \Lambda_\bullet$, is certainly exact because $\bullet\widetilde{\otimes}A$ is an exact functor. Furthermore, the isomorphisms ϕ and $\widetilde{\phi}$ ensure that the bigraded differential complex $(\Pi_\bullet\widetilde{\otimes}A) \otimes_A (\Lambda_\bullet\widetilde{\otimes}A)$ with differential $\widetilde{\delta} = \widetilde{\phi}(\delta \otimes 1)\phi$ is exact. It remains to show that $\widetilde{\delta}$ coincides with the Koszul differential on $(\Pi_\bullet\widetilde{\otimes}A)\otimes_A(\Lambda_\bullet\widetilde{\otimes}A)$. To this end consider the following diagram:

$$\begin{tikzcd}
\Pi_p\widetilde{\otimes}A\otimes_A\Lambda_q\widetilde{\otimes}A \arrow[rrr, "\widetilde{\phi}(\delta\otimes 1)\phi"] \arrow[dr, "\phi"] \arrow[ddd, "1\otimes\widetilde{\Delta}_{1,q-1}"'] & & & \Pi_{p+1}\widetilde{\otimes}A\otimes_A\Lambda_{q-1}\widetilde{\otimes}A \\
& \Pi_p\otimes\Lambda_q\widetilde{\otimes}A \arrow[r, "\delta\otimes 1"] \arrow[d, "1\otimes\Delta_{1,q-1}\otimes 1"] & \Pi_{p+1}\otimes\Lambda_{q-1}\widetilde{\otimes}A \arrow[ur, "\widetilde{\phi}"] & \\
& \Pi_p\otimes\Lambda_1\otimes\Lambda_{q-1}\widetilde{\otimes}A \arrow[r, "\cong"] \arrow[dl, "(1\otimes\widetilde{\phi})\widetilde{\phi}"] & \Pi_p\otimes\Pi_1\otimes\Lambda_{q-1}\widetilde{\otimes}A \arrow[u, "\mu\otimes 1\otimes 1"'] \arrow[dr, "(1\otimes\widetilde{\phi})\widetilde{\phi}"'] & \\
\Pi_p\widetilde{\otimes}A\otimes_A\Lambda_1\widetilde{\otimes}A\otimes_A\Lambda_{q-1}\widetilde{\otimes}A \arrow[rrr, "\cong"] & & & \Pi_p\widetilde{\otimes}A\otimes_A\Pi_1\widetilde{\otimes}A\otimes_A\Lambda_{q-1}\widetilde{\otimes}A \arrow[uuu, "\widetilde{\mu}\otimes 1"']
\end{tikzcd}\ .$$

The top and central squares commute by definition. The left-hand square commutes since

$$\begin{aligned}
(1\otimes\widetilde{\phi})\widetilde{\phi}(1\otimes\Delta_{1,q}\otimes 1)\phi(v\otimes a\otimes w\otimes b) &= (1\otimes\widetilde{\phi})\widetilde{\phi}(1\otimes\Delta_{1,q-1}\otimes 1)(v\otimes a_{(1)}w\otimes a_{(2)}b)\\
&= (1\otimes\widetilde{\phi})\widetilde{\phi}(v\otimes(a_{(1)}w)_{(1)}\otimes(a_{(1)}w)_{(2)}\otimes a_{(2)}b)\\
&= (1\otimes\widetilde{\phi})\widetilde{\phi}(v\otimes a_{(1)(1)}w_{(1)}\otimes a_{(1)(2)}w_{(2)}\otimes a_{(2)}b)\\
&= (1\otimes\widetilde{\phi})\widetilde{\phi}(v\otimes a_{(1)}w_{(1)}\otimes a_{(2)(1)}w_{(2)}\otimes a_{(2)(2)}b)\\
&= v\otimes 1\otimes a_{(1)}w_{(1)}\otimes 1\otimes a_{(2)(1)}w_{(2)}\otimes a_{(2)(2)}b\\
&= v\otimes a\otimes w_{(1)}\otimes 1\otimes w_{(2)}\otimes b\\
&= (1\otimes\widetilde{\Delta}_{1,q-1})(v\otimes a\otimes w\otimes b)\,.
\end{aligned}$$

The right-hand square commutes since

$$\begin{aligned}
(\widetilde{\mu}\otimes 1)(1\otimes\widetilde{\phi})\widetilde{\phi}(u\otimes v\otimes w\otimes a) &= (\widetilde{\mu}\otimes 1)(u\otimes 1\otimes v\otimes 1\otimes w\otimes a)\\
&= uv\otimes 1\otimes w\otimes a\\
&= \widetilde{\phi}(uv\otimes w\otimes a)\\
&= \widetilde{\phi}(\mu\otimes 1\otimes 1)(u\otimes v\otimes w\otimes a)\,.
\end{aligned}$$

Finally, it is easy to see that the bottom square commutes. Hence, the Koszul differential on $(\Pi_\bullet\widetilde{\otimes}A)\otimes_A(\Lambda_\bullet\widetilde{\otimes}A)$ coincides with $\widetilde{\phi}(\delta\otimes 1)\phi$ and is therefore exact. This concludes the proof that $(\Pi\widetilde{\otimes}A, \Lambda\widetilde{\otimes}A)$ is a Koszul pair. □

2.2.3 The affine Dynkin quivers

The primary example of a bialgebra A and Koszul pair satisfying the hypotheses of Theorem 2.2.2 is detailed in the following (cf. [MV01]):

Corollary 2.2.3. *Let V be a finite-dimensional $\mathbb{C}$-vector space and suppose that G is a finite subgroup of $\mathrm{SL}(V)$. Then $\mathrm{Sym}(V)\widetilde{\otimes}\mathbb{C}G$ and $\Lambda(V)\widetilde{\otimes}\mathbb{C}G$ form a Koszul pair, with the group algebra $\mathbb{C}G$ equipped with the diagonal coproduct:*

$$\Delta(g) = g \otimes g \quad \forall g \in G\,.$$

The relation to the affine Dynkin diagrams and their preprojective algebras is provided by the following result, proved in [CBH98]:

Theorem 2.2.4. *Let V be a two-dimensional vector space and suppose that G is a finite subgroup of $\mathrm{SL}(V)$. The twisted polynomial algebra $\mathrm{Sym}(V)\widetilde{\otimes}\mathbb{C}G$ is Morita equivalent to the preprojective algebra of an affine Dynkin quiver (see Figure 2-1).*

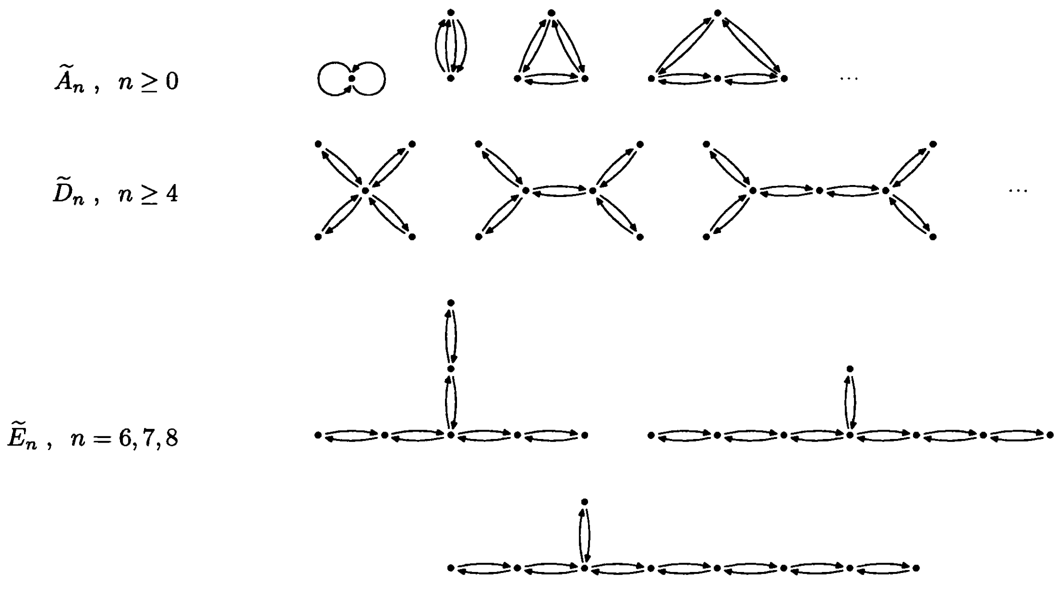


Figure 2-1: The affine Dynkin quivers

2.3 The matrix recurrence

A Koszul pair is shown to yield a solution to a particular matrix recurrence. These recurrences play an important role in the analysis of quiver algebras later in this thesis.

2.3.1 Decomposition matrices

Let V be a (finite-dimensional) bimodule for a finite-dimensional semisimple $\mathbb{C}$-algebra S. Denote the simple (left) S-modules by W_i. Then the simple S-bimodules are $W_i \otimes_{\mathbb{C}} W_j^{\vee}$, where $W_j^{\vee} = \mathrm{Hom}_S(W_j, S)$, and hence V has the isotypic decomposition

$$V \cong \bigoplus_{i,j} (W_i \otimes_{\mathbb{C}} W_j^{\vee}) \otimes_{\mathbb{C}} \mathrm{Hom}_{S,S}(W_i \otimes_{\mathbb{C}} W_j^{\vee}, V) .$$

Define the S-decomposition matrix of V by

$$M_S(V)_{i,j} = \dim \mathrm{Hom}_{S,S}(W_i \otimes_{\mathbb{C}} W_j^{\vee}, V) .$$

Notice that $M_S(\bullet)$ is an invariant of isomorphism. Moreover, $M_S(\bullet)$ is additive and multiplicative, that is,

$$\begin{aligned} M_S(A \oplus B) &= M_S(A) + M_S(B) , \\ M_S(A \otimes_S B) &= M_S(A) M_S(B) . \end{aligned}$$

2.3.2 Matrix recurrences from Koszul pairs

Let (Π, Λ) be a Koszul algebra-coalgebra pair. Then, by definition, the following collection of sequences is exact:

$$
\begin{tikzcd}[column sep=small]
\ddots & 0 \arrow[dr] & 0 \arrow[dr] & 0 \arrow[dr] & & & \\
\cdots & \Pi_0 \otimes \Lambda_4 \arrow[dr] & \Pi_0 \otimes \Lambda_3 \arrow[dr] & \Pi_0 \otimes \Lambda_2 \arrow[dr] & \Pi_0 \otimes \Lambda_1 \arrow[dr] & & \\
\cdots & \Pi_1 \otimes \Lambda_4 \arrow[dr] & \Pi_1 \otimes \Lambda_3 \arrow[dr] & \Pi_1 \otimes \Lambda_2 \arrow[dr] & \Pi_1 \otimes \Lambda_1 \arrow[dr] & \Pi_1 \otimes \Lambda_0 \arrow[dr] & \\
\cdots & \Pi_2 \otimes \Lambda_4 \arrow[dr] & \Pi_2 \otimes \Lambda_3 \arrow[dr] & \Pi_2 \otimes \Lambda_2 \arrow[dr] & \Pi_2 \otimes \Lambda_1 \arrow[dr] & \Pi_2 \otimes \Lambda_0 \arrow[dr] & 0 \\
\cdots & \Pi_3 \otimes \Lambda_4 \arrow[dr] & \Pi_3 \otimes \Lambda_3 \arrow[dr] & \Pi_3 \otimes \Lambda_2 \arrow[dr] & \Pi_3 \otimes \Lambda_1 \arrow[dr] & \Pi_3 \otimes \Lambda_0 \arrow[dr] & 0 \\
\cdots & \Pi_4 \otimes \Lambda_4 & \Pi_4 \otimes \Lambda_3 & \Pi_4 \otimes \Lambda_2 & \Pi_4 \otimes \Lambda_1 & \Pi_4 \otimes \Lambda_0 & 0 \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

But these are also exact sequences of S-bimodules for $S = \Pi_0 = \Lambda_0$, hence the exact sequences yield alternating sums of matrices upon applying $M_S(\bullet)$. Write $X_i = M_S(\Pi_i)$ and $A_i = M_S(\Lambda_i)$. Then the following collection of identities hold:

$$
\begin{aligned}
X_0 = A_0 &= I \\
X_1A_0 - X_0A_1 &= 0 \\
X_2A_0 - X_1A_1 + X_0A_2 &= 0 \\
X_3A_0 - X_2A_1 + X_1A_2 - X_0A_3 &= 0 \\
X_4A_0 - X_3A_1 + X_2A_2 + X_1A_3 - X_0A_4 &= 0 \\
&\vdots \\
\sum_{i=0}^{n}(-1)^i X_{n-i}A_i &= 0 \qquad (2.5)
\end{aligned}
$$

Suppose that a coalgebra Λ over S has been specified. Then the identities (2.5) can be interpreted as a recurrence for the matrices X_n:

$$X_n = \sum_{i=1}^{n} (-1)^{i+1} X_{n-i} A_i \,. \tag{2.6}$$

Hence, a necessary condition for the existence of an algebra Π such that (Π, Λ) is a Koszul pair, is that there is a solution to the recurrence (2.6), this solution yielding the decomposition matrices of the graded pieces of Π.

In the next chapter a particular class of recurrence is introduced and analysed.

Notes

The notion of a Koszul algebra was introduced by Priddy in [Pri70]. The standard references are now Polishchuk and Positselski [PP05] for algebras over a base field k and Beilinson, Ginzburg and Soergel [BGS96] for algebras over a semisimple base ring S. Usually Koszul duality is defined for a pair of algebras, however there are precedents for choosing an algebra-coalgebra pair (see [Val03, Introduction]).

Koszul algebras arise in many diverse areas of mathematics and are central to Manin's philosophy of quantum groups [Man87]. Recently, Vallette [Val03] has extended the notion of Koszulity to PROP-coPROP pairs, generalising the results of Ginzburg and Kapranov [GK94] on operads.

Chapter 3

The matrix recurrence

The story continues with the introduction and analysis of the SL_n matrix recurrence. Solutions are classified according to their behaviour under the recurrence; finite (bounded solution), tame (polynomial growth) or wild (exponential growth) and can be characterised by a geometric condition on eigenvalues. It is possible to classify the non-negative integer matrix solutions to the SL_2 recurrence with the finite type solutions corresponding to the Dynkin and tadpole quivers.

3.1 McKay matrices and the SL_n recurrence

A class of recurrences is introduced, solutions of which include McKay matrices for certain representations of the finite subgroups of $\mathrm{SL}_n(\mathbb{C})$.

3.1.1 McKay matrices

Let G be a finite group. Following McKay [McK80], let W_i be a complete set of representatives for the isomorphism classes of simple representations (over $\mathbb{C}$) of G. Then, for an arbitrary representation V of G and for each simple W_j, there is a semisimple decomposition

$$V \otimes W_j = \bigoplus_i M_{ij} W_i \,. \tag{3.1}$$

The resulting matrix $M = M(V)$ is called the McKay matrix of V. It is a non-negative integer matrix and, by definition, is the adjacency matrix of the McKay quiver of V.

Notice that the McKay matrix of V coincides with the decomposition matrix of the twisted bimodule $V\widetilde{\otimes}\mathbb{C}G$. Explicitly

$$
\begin{aligned}
V\widetilde{\otimes}\mathbb{C}G &= \bigoplus_j (V \otimes W_j) \otimes W_j^\vee \\
&= \bigoplus_{i,j} M_{ij} W_i \otimes W_j^\vee \, .
\end{aligned}
$$

In particular, the McKay matrix is additive and multiplicative. There will be no confusion between the notations for the McKay matrix of a left module and the decomposition matrix of a bimodule.

The following result will be needed later and appears in [McK80]:

Proposition 3.1.1. *Let V be a representation (over $\mathbb{C}$) of a finite group G and denote the McKay matrix of V by $M \in \mathrm{End}(\mathbb{C}^m)$. Then the eigenvalues of M coincide precisely with the values of the character of the dual representation V^* on each conjugacy class of G. The values of the irreducible characters of G on each conjugacy class form a corresponding eigenvector. In particular, the McKay matrix M is diagonalisable by a unitary transformation for the standard inner product on $\mathbb{C}^m$.*

Proof. Let χ_V denote the character of V and let χ_i denote the characters of the simple representations W_i. Rewriting equation (3.1) in terms of characters and evaluating on an element $g \in G$ yields

$$
\chi_V(g)\chi_j(g) = \sum_i M_{ij}\chi_i(g) = \sum_j M^T_{ji}\chi_i(g) \, .
$$

This says precisely that the values $\chi_i(g)$ form an eigenvector for M^T with eigenvalue $\chi_V(g)$. For the usual pairing on representations (which corresponds to the standard inner product on the $\mathbb{C}^m$ on which M is acting) the character table with normalised columns is a unitary transformation diagonalising M^T. Consequently, this transformation also diagonalises M, whose eigenvalues must be $\overline{\chi_V(g)} = \chi_{V^*}(g)$. □

3.1.2 The SL_n recurrence

Let V be a finite-dimensional $\mathbb{C}$-vector space. Fix a finite subgroup G of $\mathrm{SL}(V)$. Then G comes equipped with a natural representation $G \hookrightarrow \mathrm{SL}(V)$ and the McKay matrix of this representation shall be called the McKay matrix of G. Corollary 2.2.3 assures that $\mathrm{Sym}^\bullet(V)\widetilde{\otimes}\mathbb{C}G$ and $\Lambda^\bullet(V)\widetilde{\otimes}\mathbb{C}G$ form a Koszul pair. Denote the McKay matrix of $\mathrm{Sym}^i(V)$ by X_i and the McKay matrix of $\Lambda^i(V)$ by A_i. Then the computation in section 2.3.2 shows that the matrices X_i are a solution to the recurrence

$$\begin{aligned} X_0 = A_0 &= I \\ \sum_{i=0}^{m}(-1)^i X_{m-i}A_i &= 0 \,. \end{aligned} \tag{3.2}$$

The A_i possess a number of interesting properties, some of which are formalised in the following definition:

Definition 3.1.2. *Let A_i be a collection of $m \times m$ matrices satisfying the following conditions.*

1. $A_i = 0$ *for* $i > n$,
2. $A_n = I$,
3. *the A_i are simultaneously diagonalisable, in particular they are pairwise commuting.*

Recurrence (3.2) *will be called the SL_n* RECURRENCE *determined by the A_i.*

It is also interesting to notice that, in the McKay matrix situation that motivated Definition 3.1.2, the solution X_i to the recurrence (3.2) grows polynomially. This will be made more precise shortly, however, for the time being it is sufficient to note that the sum of the entries of each X_i is bounded by the dimension of $\mathrm{Sym}^i(V)\widetilde{\otimes}\mathbb{C}G$, which grows polynomially.

Choose a norm $\| \bullet \|$ on $\mathrm{End}(\mathbb{C}^m)$ (for example, the operator norm). A solution X_i to the SL_n recurrence will be called

- FINITE if there is a constant $c > 0$ such that $\|X_i\| \leq c$ for every $i \geq 0$,
- TAME if it is not finite and there is a non-constant polynomial p such that $\|X_i\| \leq p(i)$ for every $i \geq 0$,
- WILD otherwise.

In the next section, the SL_n recurrence will be analysed to find conditions on the matrices A_i that determine the behaviour of the solution.

3.2 Analysis of the SL_n recurrence

Algebraic and geometric conditions on the coefficient matrices A_i in the SL_n recurrence are obtained that determine the behaviour of the solution.

3.2.1 Algebraic conditions

Consider the SL_n recurrence determined by a collection of $m \times m$ matrices A_i subject to the conditions 1, 2 and 3 of Definition 3.1.2. In particular, the A_i are simultaneously diagonalisable, hence under such a change of basis, the recurrence decouples into m scalar recurrences of the form

$$x_{k+1} = \lambda_i(A_1)x_k - \lambda_i(A_2)x_{k-1} + \cdots + (-1)^n \lambda_i(A_{n-2}) + (-1)^{n+1} x_{k-n+1} , \tag{3.3}$$

with initial conditions

$$x_0 = 1 \quad \text{and} \quad x_{-p} = 0 \quad \forall 1 \leq p \leq n-1 \tag{3.4}$$

and where $\lambda_i(A_j)$ denotes the i^{th} eigenvalue of the matrix A_j under the ordering induced by the change of basis. The analysis in Appendix A assures that the solution to recurrence (3.3) with initial conditions (3.4) grows no faster than polynomially if and only if all the roots of the auxiliary equation lie on the unit disc. The auxiliary equation is

$$z^n - \lambda(A_1)z^{n-1} + \lambda(A_2)z^{n-2} + \cdots + (-1)^{n-1}\lambda(A_{n-1})z + (-1)^n \;=\; 0 \tag{3.5}$$

and the complex numbers $z_1, \cdots, z_n$ are roots of this equation if and only if

$$\left\{ \begin{array}{rcl} \sum\limits_i z_i & = & \lambda(A_1) , \\ \sum\limits_{i \neq j} z_i z_j & = & \lambda(A_2) , \\ \sum\limits_{i\neq j, j \neq k, i \neq k} z_i z_j z_k & = & \lambda(A_3) , \\ & \vdots & \\ \prod\limits_i z_i & = & 1 . \end{array} \right. \tag{3.6}$$

In particular, because of the last equation, all the roots of the auxiliary equation lie on the unit disc if and only if they all lie on the unit circle. Hence, the solution to recurrence (3.3) with initial conditions (3.4) grows no faster than polynomially if and only if there are unit complex numbers $z_1, \ldots, z_n$ satisfying the above conditions, that is, the auxiliary equation (3.5) is the characteristic equation of a matrix $B \in \mathrm{SU}(n)$. In particular, each eigenvalue λ_i of A_1 must be the trace of a matrix $B_i \in \mathrm{SU}(n)$.

The remainder of this chapter is devoted to analysing this last condition: in order that the solution to the SL_n recurrence is finite or tame, every eigenvalue of A_1 must be the trace of a matrix in $\mathrm{SU}(n)$. This condition is necessary but by no means sufficient. However, for such an A_1 it is always possible to construct a sequence of matrices A_i for which the solution to the corresponding SL_n recurrence is finite or tame: it is sufficient to fix a diagonalising change of basis for A_1 and matrices B_i whose trace yields the eigenvalue λ_i.

The matrix A will be said to be n-TAME (respectively n-FINITE, n-WILD) if there is a collection of matrices A_i with $A_1 = A$ and such that the SL_n recurrence determined by the A_i has a tame (finite, wild) solution. If A is an n-tame (n-finite, n-wild) non-negative integer matrix then the quiver defined by A will be called n-TAME (n-FINITE, n-WILD). These three properties of the matrix A are not necessarily mutually exclusive.

Proposition 3.2.1 casts a cursory glance back towards the subject that motivated the study of tame solutions to the SL_n recurrence.

Proposition 3.2.1. *The McKay quiver of a finite subgroup $G \subset \mathrm{SU}(n)$ is n-tame.*

Proof. Let $\rho \colon G \hookrightarrow \mathrm{SU}(n)$ denote the given representation and let A be the McKay matrix of ρ. Let ρ^* denote the corresponding dual representation. The proof is in three parts. First a number of scalar recurrences are identified, each of which is the characteristic equation of a matrix $B_i \in \mathrm{SU}(n)$. Secondly, the scalar recurrences are combined to form an SL_n recurrence with $A_0 = I$ and $A_1 = A$. Finally, it is shown that the solution to this recurrence is tame and not finite.

Choose an element g_i in each conjugacy class of G and define $B_i = \rho^*(g_i) \in \mathrm{SU}(n)$. For each $i = 1, \ldots, m$ write the characteristic polynomial of B_i as

$$P_i(x) = \sum_{j=1}^{n} c_j^{(i)} x^{n-j} \ .$$

Of course, $c_0^{(i)} = 1$ by definition, $c_n^{(i)} = (-1)^n \det(B_i) = (-1)^n$ and $c_1^{(i)} = (-1)\ \mathrm{tr}(B_i)$. The scalar recurrence with this auxiliary equation is

$$\sum_{j=1}^{n} c_j^{(i)} x_{r-j} \ ,$$

the solution of which can therefore grow no faster than polynomially. The next step is to combine these scalar recurrences in an appropriate way. First define the diagonal matrices $D_j = \mathrm{diag}((-1)^j c_j^{(1)}, \ldots (-1)^j c_j^{(m)})$. Notice that $D_0 = I = D_n$ and $D_1 = \mathrm{diag}(\mathrm{tr}(B_1), \ldots, \mathrm{tr}(B_m))$. By Proposition 3.1.1, the eigenvalues of the McKay matrix

A are precisely $\operatorname{tr}(B_i)$ for each i and there is a unitary transformation U such that $D_1 = U^*AU$. Define

$$A_j = UD_jU^*$$

for $j = 0, \ldots, n$ and define $A_j = 0$ for $j > n$. Then the matrices A_j are the coefficients of an SL_n recurrence

$$\sum_{j=0}^{r} (-1)^j X_{r-j} A_j = 0 \, .$$

By construction (and due to the analysis in this section) the solution to this SL_n recurrence must be finite or tame.

The only remaining statement to prove is that the solution is tame and not finite. By Proposition 3.1.1, n is an eigenvalue for A. The corresponding scalar recurrence reduces to

$$\begin{aligned} x_{r+1} &= nx_r - \binom{n}{2} x_{r-1} + \binom{n}{3} x_{r-2} + \ldots \\ &\quad +(-1)^n \binom{n}{n-1} x_{r-n+1} + (-1)^{n+1} x_{r-n} \end{aligned} \tag{3.7}$$

for $r > n$. This is because the only partition of n into a sum of n unit complex numbers is $n = \overbrace{1 + \cdots + 1}^{n \text{ times}}$ and hence the eigenvalue of A_k associated with α is $\binom{n}{k}$. The auxiliary equation of the recurrence (3.7) is

$$(z-1)^n = 0 \, .$$

By Lemma A.2.1, $x_r = r^{n-1}$ is a solution of (3.7). Furthermore, the lemma assures that this is the optimal growth the recurrence can produce since the auxiliary equations (3.5) of the recurrences (3.3) possess a root with multiplicity at most n. As a consequence of Proposition A.3.1, the unique solution to the SL_n recurrence with coefficients A_j defined above must also grow at this rate. □

3.2.2 Geometric conditions

In this section, the condition that each eigenvalue λ_i of a matrix A is the trace of a matrix $B_i \in \mathrm{SU}(n)$ is investigated further. In particular, it is established that the set of values for the trace function $\operatorname{tr}\colon \mathrm{SU}(n) \to \mathbb{C}$ coincides precisely with the set of points in the interior and on the boundary of the n-cusped hypocycloid.

The n-cusped hypocycloid is defined as the locus traced by a point on a circle of radius 1 rolling inside a larger circle of radius n. The pictures below show the 3,4,5 and 6-cusped hypocycloids (scaled for comparison):

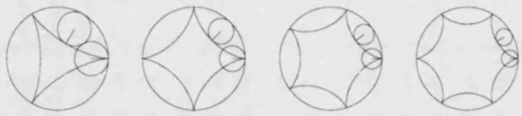


Figure 3-1: The 3,4,5 and 6-cusped hypocycloids.

Proposition 3.2.2. *Let $n \geq 3$. The n-cusped hypocycloid is strictly star-shaped, that is, each radial half-line from the origin intersects the hypocycloid at precisely one point. In particular, it is a simple closed curve.*

Proof. For $\theta \in [0, 2\pi)$ the n-cusped hypocycloid can be parameterised by

$$\begin{aligned} \lambda(\theta) &= (n-1)e^{i\theta} + e^{-i(n-1)\theta} \\ &= \overbrace{(n-1)\cos\theta + \cos(n-1)\theta}^{v} + i\,\overbrace{((n-1)\sin\theta - \sin(n-1)\theta}^{u}\ . \end{aligned}$$

Hence

$$\operatorname{Arg}(\lambda(\theta)) = \arctan\frac{u}{v} + \text{constant}\ .$$

Differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\operatorname{Arg}(\lambda(\theta))\right) = \frac{1}{1+\frac{u^2}{v^2}}\left(\frac{v\frac{\mathrm{d}u}{\mathrm{d}\theta} - u\frac{\mathrm{d}v}{\mathrm{d}\theta}}{v^2}\right)\ .$$

Now

$$\begin{aligned} v\frac{\mathrm{d}u}{\mathrm{d}\theta} - u\frac{\mathrm{d}v}{\mathrm{d}\theta} &= [(n-1)\cos\theta + \cos(n-1)\theta][(n-1)\cos\theta - (n-1)\cos(n-1)\theta] \\ &\quad + [(n-1)\sin\theta - \sin(n-1)\theta][(n-1)\sin\theta + (n-1)\sin(n-1)\theta] \\ &= (n-1)^2\cos^2\theta - (n-1)\cos^2(n-1)\theta \\ &\quad + [(n-1) - (n-1)^2]\cos\theta\cos(n-1)\theta \\ &\quad + (n-1)^2\sin^2\theta - (n-1)\sin^2(n-1)\theta \\ &\quad + [(n-1)^2 - (n-1)]\sin\theta\sin(n-1)\theta \\ &= (n-1)^2 - (n-1) + [(n-1) - (n-1)^2]\cos n\theta \\ &= (n-1)(n-2)(1-\cos n\theta) \geq 0 \quad \forall n \geq 3\ , \end{aligned}$$

with equality at the finite number of points corresponding to $\theta = \frac{2\pi k}{n}$, $k = 0, \ldots, n-1$. Hence $\mathrm{Arg}(\lambda(\theta))$ is strictly increasing except at a finite number of points. $\square$

Theorem 3.2.3. *The possible values for the trace of a matrix $M \in SU(n)$ are precisely the points in the interior and on the boundary of the n-cusped hypocycloid.*

Proof. For $n = 2$ the line segment [-2,2] is recovered in each case. Assume $n \geq 3$.

Let $V = \{\theta_1, \ldots, \theta_n \in [0, 2\pi) \mid \sum_{i=1}^{n} \theta_i = 0 \pmod{2\pi}\}$. Then the possible traces of a matrix $M \in \mathrm{SU}(n)$ are described by the function

$$\begin{aligned} \lambda : V &\longrightarrow \mathbb{C} \\ (\theta_1, \ldots, \theta_n) &\longmapsto \sum_{p=1}^{n} e^{i\theta_p} . \end{aligned}$$

Clearly, λ is invariant under permutations of the angles θ_p. It may be convenient occasionally to think of λ as a function

$$\begin{aligned} \widetilde{\lambda} : [0, 2\pi]^{n-1} &\longrightarrow \mathbb{C} \\ (\theta_1, \ldots, \theta_{n-1}) &\longmapsto e^{-i\sum_{q=1}^{n-1}\theta_q} + \sum_{p=1}^{n-1} e^{i\theta_p} . \end{aligned}$$

It is clear therefore that the image of λ is a compact set K.

The geometric picture for λ is as follows: n rods of unit length are placed end to end, starting at the origin and terminating at λ, the angle of the p^{th} rod to the x-axis being θ_p. The locus traced by the terminal point λ as the θ_p vary is the set of all possible traces of a matrix $M \in \mathrm{SU}(n)$.

The locus $L := \{\lambda(\theta_1, \ldots, \theta_n) \mid \theta_1 = \theta_2 = \cdots = \theta_{n-1}\}$ describes the n-cusped hypocycloid and hence, if any $n-1$ of the angles θ_p are equal then the corresponding λ must lie on this hypocycloid. It will be shown that all other configurations must lie in the interior of the hypocycloid and that all points in the interior of the hypocycloid are in the image of λ.

First, it is shown that

$$\lambda^{-1}(\partial K) \subset \Delta = \{(\theta_1, \ldots, \theta_n) \mid n-1 \text{ of the components are equal}\} .$$

Let $\lambda = \lambda(\theta_1, \dots, \theta_n)$ for $(\theta_1, \dots, \theta_n) \notin \Delta$. It will be seen that λ lies on (but is not an endpoint of) a short line, which is part of a 1-parameter family of lines in $\lambda(V)$. This 1-parameter family of lines sweeps out a neighbourhood of $\lambda(\theta_1, \dots, \theta_n)$. Hence, $\lambda(\theta_1, \dots, \theta_n)$ will be in the interior of the compact K. There are two ways in which $(\theta_1, \dots, \theta_n)$ can fail to be in Δ, hence there are two cases to consider:

Case 1: Three of the θ_i are distinct. Reordering if necessary, assume without loss of generality that θ_1, θ_2 and θ_3 are distinct. Define $\phi_{12} = \frac{1}{2}(\theta_1 + \theta_2)$ and $\phi_{23} = \frac{1}{2}(\theta_2 + \theta_3)$. Now, $\lambda(\theta_1, \theta_2, \theta_3, \dots)$ lies on (but is not an endpoint of) the lines

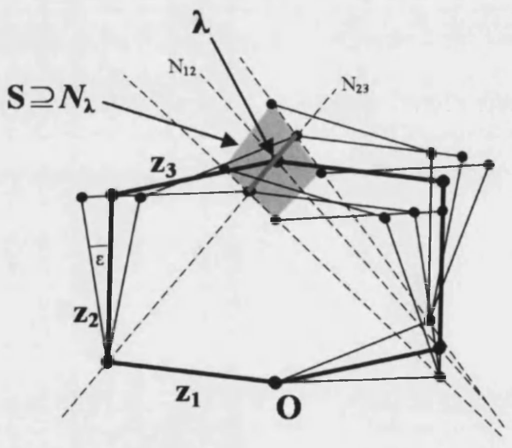
$$\begin{aligned} N_{12} &= \{\lambda(\phi_{12} + t, \phi_{12} - t, \theta_3, \dots) \mid t \in [0, \pi]\} , \\ N_{23} &= \{\lambda(\theta_1, \phi_{23} + s, \phi_{23} - s, \dots) \mid s \in [0, \pi]\} . \end{aligned}$$

Choose $\epsilon > 0$ sufficiently small that the sets

$$\{\lambda(\theta_1 + t, \theta_2 - t, \theta_3, \dots) \mid t \in [-\epsilon, \epsilon]\} \quad \& \quad \{\lambda(\theta_1, \theta_2 + s, \theta_3 - s, \dots) \mid s \in [-\epsilon, \epsilon]\}$$

do not contain the endpoints of the lines N_{12} and N_{23}. Then $\lambda = \lambda(\theta_1, \theta_2, \theta_3, \dots)$ is an interior point of the set $S = \{\lambda(\theta_1 + t, \theta_2 - t + s, \theta_3 - s, \dots) \mid s, t \in [-\epsilon, \epsilon]\}$.

The picture described is the following:



Case 2: There are only two distinct angles $\alpha \neq \beta$ and at least two of each. Reordering if necessary, assume without loss of generality

$$\theta_1 = \alpha = \theta_2 \ , \quad \theta_3 = \beta = \theta_4 \ .$$

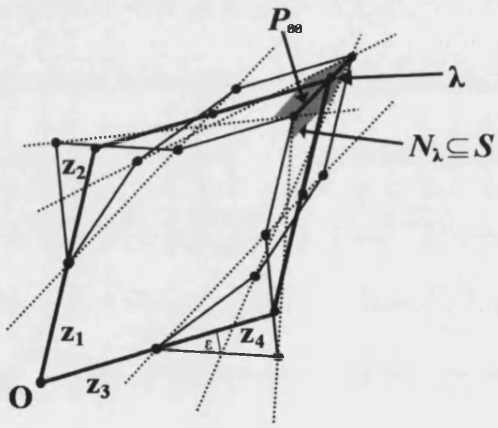
For $\epsilon > 0$ small, consider the sets

$$\begin{aligned} S_1 &= \{\lambda(\alpha + r, \alpha - r + t, \beta - t, \beta, \dots) \mid r \in [0, \epsilon] \ , \ t \in [-\epsilon, \epsilon]\} \ , \\ S_2 &= \{\lambda(\alpha, \alpha + t, \beta - t + s, \beta - s, \dots) \mid s \in [0, \epsilon] \ , \ t \in [-\epsilon, \epsilon]\} \ , \\ S &= S_1 \cup S_2 \ . \end{aligned}$$

It is claimed that $\lambda = \lambda(\alpha, \alpha, \beta, \beta, \dots)$ is an interior point of S. To see this notice that for fixed r and s,

$$P_{rs} = \{\lambda(\alpha + r, \alpha - r + t, \beta - t + s, \beta - s, \dots) \mid t \in [-\epsilon, \epsilon]\}$$

is a straight line and therefore S_1 and S_2 describe 1-parameter families of straight lines. The endpoints of the line P_{00} (which is the common edge of these two sets) move along 4 fixed straight lines. The point λ lies on (but is not an endpoint of) P_{00}. As r and s increase from 0 to ϵ, the image of the line P_{00} moves in such a way that λ is left in the interior of the set S.



It has been shown therefore that

$$\lambda^{-1}(\partial K) \subset \{(\theta_1, \dots, \theta_n) \mid n - 1 \text{ of the components are equal} \}$$

as required. Hence, the boundary of the set K lies on the n-cusped hypocycloid. Now the hypocycloid is strictly star-shaped and K has interior points (the point $x_0 = \lambda(0, \frac{\pi}{2}, \frac{3\pi}{2}, 0, \dots, 0)$ is an interior point of K, for example, since three of the angles are distinct). The compact K must therefore be the entire interior and boundary of the n-cusped hypocycloid. Suppose for contradiction there is a point $x \in K$ in the exterior of the hypocycloid curve. Consider the radial half-line $H := \{tx \mid t \in \mathbb{R}_{\geq 0}\}$, starting at the origin and passing through x. Now, K is compact, hence there is a $t_0 \geq 1$ such that $t_0 = \max(\{t \geq 0 \mid tx \in K\})$. Then t_0 is a boundary point of K which contradicts the fact that the boundary of K lies on the hypocycloid curve. Now suppose there is a point $x \notin K$ in the interior of the hypocycloid curve, or on the curve itself. Consider the path $P(t) = \mathbb{I}_{t\in[0,1]}(1-t)x + \mathbb{I}_{t\in[1,2]}(t-1)x_0$, which does not leave the region bounded by the hypocycloid curve. Then there is a $t_1 > 0$ such that $t_1 = \min(\{t \in [0,2] \mid P(t) \in K\})$. Such a t_1 is a boundary point of K in the strict interior of the region bounded by the hypocycloid curve, contradicting the fact that the boundary of K lies on the hypocycloid curve. □

3.3 The SL_2 and SL_3 recurrences

The remainder of this chapter takes a closer look at the SL_2 and SL_3 recurrences. The classification of the finite and tame quivers for the SL_2 recurrence is discussed. A collection of quivers from rational boundary conformal field theory is introduced and appear to yield finite solutions to the SL_3 recurrence.

3.3.1 Introducing a little more structure

To facilitate further analysis of the SL_n recurrences a little more structure is introduced. One of the key properties of the matrices A_i, enabling a straightforward analysis of the recurrence, was that they are simultaneously diagonalisable. In fact, for McKay matrices more is true. There is a natural pairing on representations inducing the standard inner product on $\mathbb{C}^m$. Proposition 3.1.1 assures that the diagonalisation of the A_i may be chosen to be unitary with respect to this inner product. The character table, conveniently normalised, is one such unitary transformation.

Henceforth, suppose that the coefficient matrices A_i in the SL_n recurrence satisfy the stronger condition

3′. the A_i are simultaneously diagonalisable *by a unitary transformation.*

In particular the matrices A_i must be normal. This structure has been seen to arise naturally in the McKay matrix case.

Proposition 3.3.1. *If the solution to the* SL_n *recurrence determined by the* A_i *is tame or finite, then* $A_i^* = A_{n-i}$*, where* $*$ *denotes the adjoint with respect to the inner product.*

Proof. The A_i are simultaneously diagonalisable by a unitary transformation U, hence for each i,

$$A_i = U^* \mathrm{diag}(\lambda_1(A_i), \ldots, \lambda_m(A_i))U \ .$$

Furthermore, the solution to the recurrence grows no faster than polynomially, hence for each $j = 1, \ldots, m$ there exist unit complex numbers $z_1^{(j)}, \ldots, z_n^{(j)}$ satisfying

$$\begin{cases} \sum_a z_a^{(j)} & = & \lambda_j(A_1) \, , \\ \sum_{a \neq b} z_a^{(j)} z_b^{(j)} & = & \lambda_j(A_2) \, , \\ \sum_{a\neq b, b\neq c, a\neq c} z_a^{(j)} z_b^{(j)} z_c^{(j)} & = & \lambda_j(A_3) \, , \\ & \vdots & \\ \prod_a z_a^{(j)} & = & 1 \, . \end{cases}$$

Notice that for $I \subset \{1, \ldots, n\}$,

$$\left(\prod_{a \in I} z_a^{(j)} \right)^* = \prod_{b \in \{1,\ldots,n\} \setminus I} z_b^{(j)} \, .$$

Hence

$$\lambda_j(A_i)^* = \lambda_j(A_{n-i})$$

and therefore

$$\begin{aligned} A_i^* &= (U^* \mathrm{diag}(\lambda_1(A_i), \ldots, \lambda_m(A_i)) U)^* \\ &= U^* \mathrm{diag}(\lambda_1(A_i)^*, \ldots, \lambda_m(A_i)^*) U \\ &= U^* \mathrm{diag}(\lambda_1(A_{n-i}), \ldots, \lambda_m(A_{n-i})) U \\ &= A_{n-i} \, . \end{aligned}$$

□

In particular, a 2-tame or 2-finite matrix must be self-adjoint.

The remainder of this chapter is primarily concerned with non-negative integer matrices A_i. Thus A_i^* coincides with A_i^T for the standard inner product on $\mathbb{C}^m$.

3.3.2 The SL_2 recurrence

Consider the SL_2 recurrence

$$\begin{aligned} X_{i+1} = X_i A - X_{i-1} \, , & \qquad i \geq 1 \, , \\ X_0 = I \, , \ X_1 = A \, . & \end{aligned}$$

Then the solution X_i is tame or finite if and only if A is self-adjoint and its eigenvalues lie in the interval $[-2, 2]$. It is possible to be more precise. If A has an eigenvalue equal to ± 2, then Lemma A.2.1 ensures that the solution is tame, since the auxiliary equation has a double root. If all the eigenvalues of A lie strictly in the interval $(-2, 2)$ then Lemma A.2.1 ensures that the sequence $\|X_i\|$ is bounded. The above discussion is summarised in the following theorem:

Theorem 3.3.2. *Let A be a self-adjoint matrix. Define the sequence of matrices $\{X_i\}_{i=0}^{\infty}$ as the solution of the recurrence*

$$X_{i+1} = X_i A - X_{i-1} \, , \qquad i \geq 1 \, ,$$

with initial conditions $X_0 = I$ and $X_1 = A$. The following result holds for the entries of the matrices X_i:

- *if each eigenvalue of A lies in the open interval $(-2, 2)$ then the sequence is bounded;*
- *if each eigenvalue of A lies in the closed interval $[-2, 2]$ and if at least one eigenvalue is equal to ± 2 then the sequence grows linearly;*
- *if at least one eigenvalue of A lies outside the closed interval $[-2, 2]$ then the sequence grows exponentially.*

Notice that the three conditions are mutually exclusive. Consider the following quadratic form:

$$q(A) = 4I - A^2 \, .$$

Then the classification can be rewritten in the following way:

Theorem 3.3.3. *Let A be a self-adjoint matrix. Then*

- *A is 2-finite if $q(A)$ is positive definite,*
- *A is 2-tame if $q(A)$ is positive semi-definite (but not definite),*
- *A is 2-wild if $q(A)$ is not positive semi-definite.*

The remainder of this section discusses the classification of 2-tame and 2-finite quivers for the standard inner product on $\mathbb{C}^m$. Consider therefore, a symmetric non-negative integer matrix A. It suffices to consider only connected quivers and since A is symmetric these quivers must be strongly connected. The adjacency matrix A of a strongly connected quiver is irreducible and the spectral radius of A is necessarily an eigenvalue of A. Hence, A is 2-finite if and only if A has maximal eigenvalue strictly less than 2 and A is 2-tame if and only if A has maximal eigenvalue equal to 2. (In the language of quadratic forms, for A a symmetric non-negative integer matrix, it is equivalent to consider the quadratic form $2I - A$ in place of $4I - A^2$.)

The connected symmetric quivers with maximal eigenvalue strictly less than 2 are well-known (see [GdlHJ89] for example). They consist of the (finite) Dynkin quivers and the tadpole quivers shown in Figure 3-2 below.

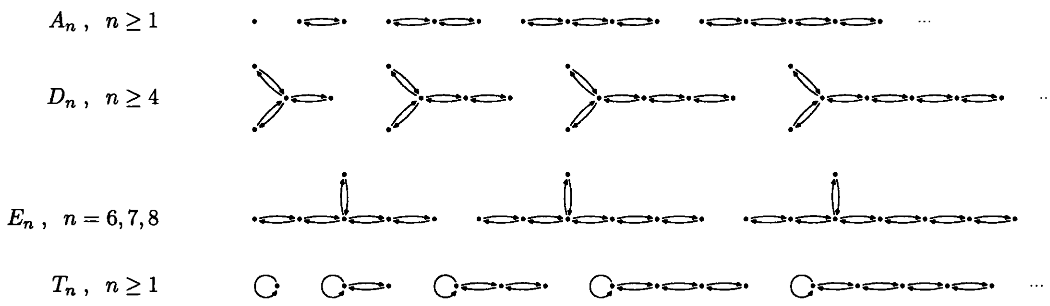


Figure 3-2: The finite Dynkin and tadpole quivers.

The connected symmetric quivers with maximal eigenvalue equal to 2 correspond to the affine Dynkin quivers and the affine tadpole quivers in Figure 3-3 below.

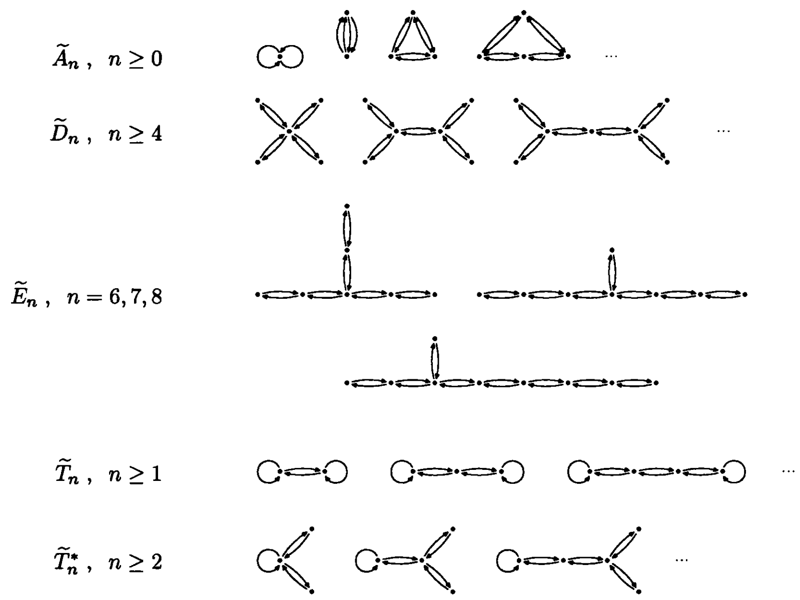


Figure 3-3: The affine Dynkin and affine tadpole quivers

These classifications can be obtained by showing that the maximal eigenvalue of a symmetric strongly connected quiver is monotonic with respect to subquivers. It then remains to show that every symmetric strongly connected quiver is either in one of the lists or possesses a strict subquiver in the second list.

The fact that the affine Dynkin quivers are 2-tame is, of course, no surprise - they are the McKay quivers of the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Moreover, it will be seen later that every symmetric quiver that is not 2-finite supports a Koszul algebra-coalgebra pair. The main focus of the thesis will, however, be on an algebraic structure closely related to Koszul duality for the 2-*finite* quivers. *Almost Koszul duality* ([BBK02]) will be shown to be intimately connected to lattice models for rational boundary conformal field theories.

A key result on this path is the following, pointed out by Di Francesco and Zuber [DFZ90a]:

Proposition 3.3.4. *Let Q be a Dynkin A, D, E or a tadpole quiver and let h denote the Coxeter number of Q (the Coxeter number of T_n is $2n-2$). Let A be the adjacency matrix of Q. The solution $\{X_i\}$ to the SL_2 recurrence defined by A satisfies $X_{h-1}=0$.*

Proof. The recurrence is diagonalisable and therefore it suffices to prove the result for the scalar recurrence defined by each eigenvalue of Q. Moreover, the spectrum of Q is a subset of the spectrum of A_{h-1} (see [GdlHJ89] and [Zub02]). Let λ be an eigenvalue for A_{h-1}. Then λ is of the form $\lambda = 2\cos\frac{\pi k}{h}$, where $1 \leq k \leq h-1$ is a Coxeter exponent. The SL_2 scalar recurrence defined by λ has solution $x_{r-1} = \frac{\sin\frac{\pi k r}{h}}{\sin\frac{\pi k}{h}}$. Thus $x_{h-1}=0$ as claimed. $\square$

There is an immediate corollary, returning to describe the nature of the bounded recurrence for the 2-finite quivers:

Corollary 3.3.5. *The solution to the SL_2 recurrence defined by the adjacency matrix of a 2-finite quiver is $2(h-1)$-periodic.*

3.3.3 The SL_3 recurrence

Consider the SL_3 recurrence

$$X_{i+1} = X_i A_1 - X_{i-1} A_2 + X_{i-2}\ , \quad i \geq 2\ ,$$
$$X_0 = I\ , \quad X_1 = A_1\ , \quad X_2 = A_1^2 - A_2\ .$$

Recall that A_1 must be normal and if A_1 is 3-finite or 3-tame then it has been shown that $A_2 = A_1^*$ and the eigenvalues of A_1 must lie on the boundary or in the interior of the 3-cusped hypocycloid (DELTOID). The following theorem makes the classification more precise:

Theorem 3.3.6. *Let A be a normal matrix, that is, $AA^* = A^*A$. Define the sequence of matrices $\{X_i\}_{i=0}^{\infty}$ as the solution of the recurrence*

$$X_{i+1} = X_i A - X_{i-1} A^* + X_{i-2} \ , \qquad i \geq 2 \ ,$$

with initial conditions $X_0 = I$, $X_1 = A$ and $X_2 = A^2 - A^*$. The following result holds for the entries of the matrices X_i:

- *if each eigenvalue of A lies in the interior of the deltoid then the sequence is bounded;*
- *if at least one eigenvalue of A lies on the boundary of the deltoid and if the other eigenvalues of A lie in the interior of the deltoid then the sequence grows polynomially;*
- *if at least one eigenvalue of A lies outside the deltoid then the sequence grows exponentially.*

Proof. The discussion is identical to that for Theorem 3.3.2. □

The precise nature of the polynomial growth will depend on the location of the eigenvalues on the boundary of the deltoid. If there is an eigenvalue at a cusp then the growth will be quadratic; otherwise it will be linear.

The equation of the points inside the deltoid is

$$8r^3 \cos 3\theta - r^4 - 18r^2 + 27 \geq 0 \ ,$$

which in complex coordinates becomes

$$4\lambda^3 + 4\overline{\lambda}^3 - \lambda^2\overline{\lambda}^2 - 18\lambda\overline{\lambda} + 27 \geq 0 \ .$$

Consider therefore, the following quadratic form:

$$q(A) = 4A^3 + 4(A^*)^3 - A^2(A^*)^2 - 18AA^* + 27I \ .$$

Then the classification can be restated in the following way:

Theorem 3.3.7. *Let A be a normal matrix. Then*

- *A is 3-finite if $q(A)$ is positive definite,*
- *A is 3-tame if $q(A)$ is positive semi-definite (but not definite),*
- *A is 3-wild if $q(A)$ is not positive semi-definite.*

It is interesting that the analysis returns to the requirement that a particular quadratic form be positive (semi-)definite. Whilst the quadratic form $2I - A$ (for self-adjoint A) arises in a number of areas of mathematics, the author is not aware of the occurrence in the literature of the quadratic form

$$q(A) = 4A^3 + 4(A^*)^3 - A^2(A^*)^2 - 18AA^* + 27I ,$$

although the deltoid appears in this context in [DFZ90b]. A classification of the 3-finite and 3-tame quivers for the standard inner product on $\mathbb{C}^m$ has not been attempted by the author and there are examples of 3-wild subquivers of 3-tame quivers. However, Di Francesco and Zuber [Zub02, DFZ90b] provide a list of quivers for the $\widehat{\mathfrak{sl}}_3$ rational boundary conformal field theories, which have normal adjacency matrices and are claimed to be 3-finite ([DFZ90b, DF92]). The Di Francesco-Zuber list of quivers is reproduced in Figures 3-4 and 3-5 by kind permission of J.-B. Zuber.

The coloured vertices on some of the quivers encode the direction of the edges (black $\rightarrow$ grey $\rightarrow$ white, for example). These graphs were found empirically, subject to a number of constraints ([DFZ90b]). In particular, the spectrum of each quiver (the set of eigenvalues for the adjacency matrix) is constrained to be a subset of the spectrum of one of the $\mathcal{A}^{(h)}$ quivers. Notice therefore that all these graphs are 3-finite if the $\mathcal{A}^{(h)}$ graphs are 3-finite.

In Chapter 7 a framework is established to show that the Di Francesco-Zuber quivers support almost Koszul algebra-coalgebra pairs. The key result needed here is proved in Section 3.4 (but is an immediate consequence of one of the $\widehat{\mathfrak{sl}}_3$ quiver axioms ([DFZ90a])):

Theorem 3.3.8. *Let $\{X_i\}$ be the solution to the* SL_3 *recurrence*

$$\begin{aligned} X_{i+1} &= X_i A - X_{i-1}A^T + X_{i-2} , \quad i \geq 2 , \\ X_0 &= I , \quad X_1 = A , \quad X_2 = A^2 - A^T , \end{aligned}$$

where A is the adjacency matrix of the Di Francesco-Zuber $\mathcal{A}^{(h)}$ quiver. Then $X_{h-2} = 0$.

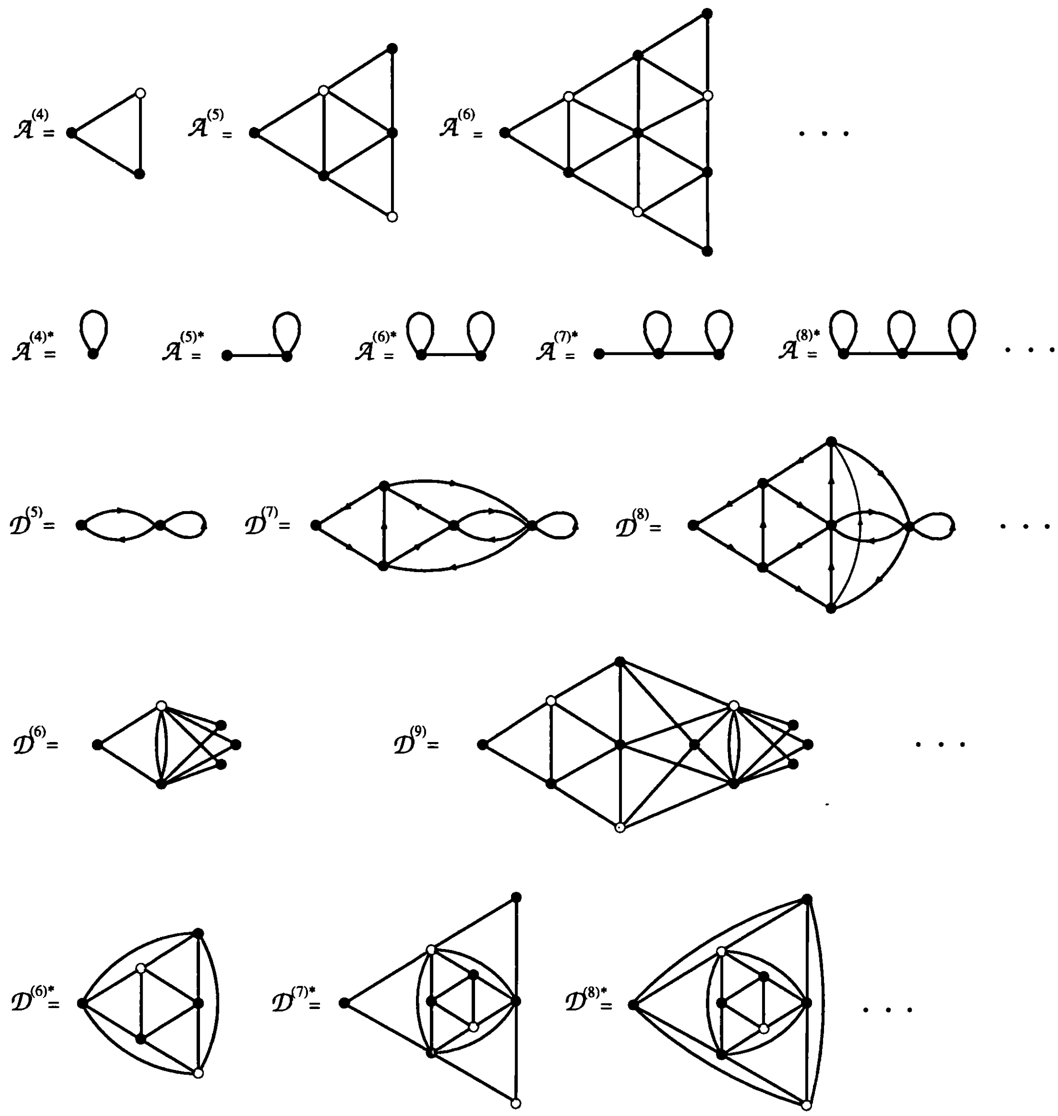


Figure 3-4: The Di Francesco-Zuber graphs of $\widehat{\mathfrak{sl}}_3$.

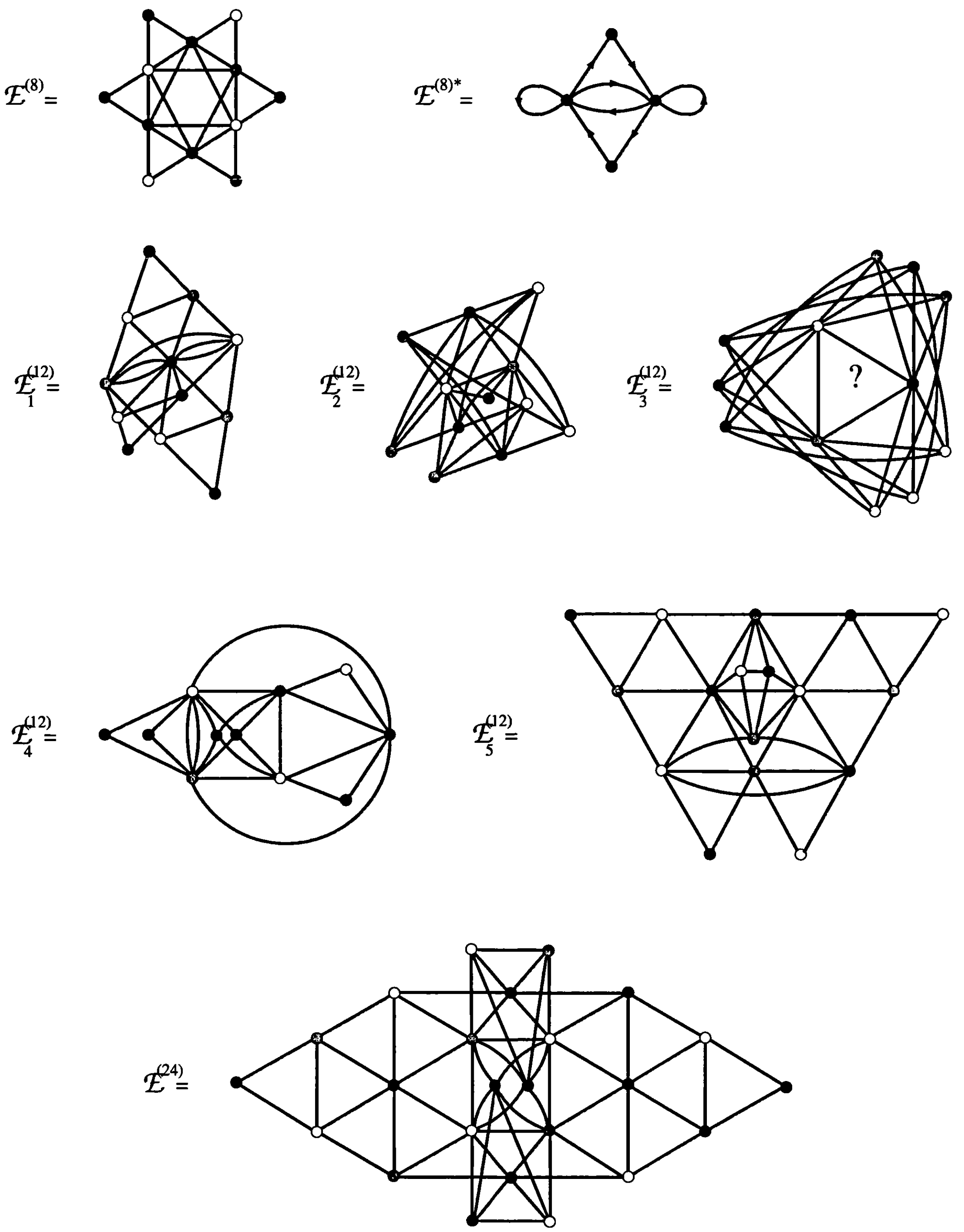


Figure 3-5: The Di Francesco-Zuber graphs of $\widehat{\mathfrak{sl}}_3$ (cont.).

3.3.4 The SL_n recurrence for $n \geq 4$

Some of the analysis for the SL_3 recurrence can be repeated for the higher SL_n recurrences, but these results become restatements of the algebraic analysis and do not shed much light on the situation. This is largely due to the fact that for $n \geq 4$, the matrix A_1 no longer determines the other coefficient matrices in the SL_n recurrence.

For example, if A_1 is a 4-tame or 4-finite matrix then its eigenvalues must lie on the boundary or in the interior of the 4-cusped hypocycloid (ASTROID). However, it is no longer the case that the precise location of the eigenvalues determines the behaviour of the recurrence. If the eigenvalues of A_1 lie in the interior of the astroid, then there exists an A_2 for which the solution of the SL_4 recurrence is tame and an A_2 for which the solution is finite. This results from the fact that there is the choice of expressing any interior point of the astroid as the sum of 4 distinct unit complex numbers, or as the sum of 4 unit complex numbers with at least one repetition. This situation occurs for each of the n-cusped hypocycloids with $n \geq 4$.

3.4 A Cayley-Hamilton theorem for quivers

The aim of this section is to prove that the adjacency matrix A for a quiver satisfies the fusion problem it defines, provided A is diagonalisable and there are sufficiently many eigenvectors that do not vanish on some vertex. In particular, it is verified that the Di Francesco-Zuber $\mathcal{A}^{(h)}$ quivers satisfy the truncated SU(3) fusion rule ([DFZ90a]). As a result, it is shown that the solution $\{X_i\}$ to the SL_3 recurrence defined by the adjacency matrix of the $\mathcal{A}^{(h)}$ quiver satisfies $X_{h-2} = 0$.

For the remainder of this chapter maps are declared to compose left-to-right, coinciding with the action of the adjacency matrix on vectors.

3.4.1 Eigenvectors for quivers

Let $Q = (Q_0, Q_1)$ be a quiver and denote the adjacency matrix by A. Fix a complex number λ and consider the following problem:

> Find a complex-valued function $\phi\colon Q_0 \to \mathbb{C}$ satisfying, for every vertex $i \in Q_0$,
>
> $$\sum_{j \to i} \phi(j) = \lambda \phi(i)\ .$$

This is the (λ-)eigenvector problem for Q. For λ fixed, the set of solutions is precisely the eigenspace $E(\lambda) = \mathrm{Ker}(A - \lambda\ \mathrm{id}) \subset \mathbb{C}^{Q_0}$.

Definition 3.4.1. *An* EIGENVECTOR *for Q with* EIGENVALUE λ *is a non-trivial solution to the λ-eigenvector problem.*

Of course, the notion of an eigenvector for Q coincides precisely with the notion of an eigenvector for A.

3.4.2 Fusion rules

Let Q be a quiver and fix a matrix $A \in \mathrm{End}(\mathbb{C}^n)$ for some $n \in \mathbb{N}$. Consider the following FUSION PROBLEM:

Find a function $\Phi\colon Q_0 \longrightarrow \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ satisfying, for each vertex $i \in Q_0$,

$$\sum_{j \to i} \Phi(j) = \Phi(i)A\ . \tag{3.8}$$

If A is chosen to be the adjacency matrix of the quiver Q, then this is a matrix version of the eigenvector problem. Additional constraints can be imposed on the fusion problem. The remainder of this section will be concerned with the following:

Fix a vertex $i \in Q_0$ and an arbitrary matrix $B \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$. Find a solution Φ to the fusion problem satisfying $\Phi(i) = B$.

The following result is an analogue of the Cayley-Hamilton Theorem:

Theorem 3.4.2. *Let Q be a quiver with a diagonalisable adjacency matrix $A \in \mathrm{End}(\mathbb{C}^{Q_0})$. Fix a vertex $i \in Q_0$ and suppose that for each eigenvalue of Q there exists at least one corresponding eigenvector that does not vanish on the vertex i. Then, for arbitrary $B \in \mathrm{End}(\mathbb{C}^{Q_0})$, there is a solution Φ to the fusion problem* (3.8) *satisfying $\Phi(i) = B$.*

The motto here is that "Q satisfies its own fusion rule".

Proof. Notice that it is sufficient to find a solution Φ satisfying $\Phi(i) = I$, since then $B\Phi$ solves the specified fusion problem.

Let U be a diagonalising change of basis for A and use this to diagonalise the fusion problem:

$$\begin{aligned} \sum_{j \to k} U^{-1}\Phi(j)U &= U^{-1}\Phi(k)UU^{-1}AU \\ &= (U^{-1}\Phi(k)U)\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\ . \end{aligned}$$

The first step in the proof is to identify a solution to the diagonalised fusion problem:

$$\sum_{j\to k} \Psi(j) = \Psi(k)\mathrm{diag}(\lambda_1, \ldots, \lambda_n) , \tag{3.9}$$

where $\Psi(i) = I$. By hypothesis, for each eigenvalue λ_k there exists a corresponding eigenvector ϕ_k satisfying $\phi_k(i) \neq 0$. By rescaling ϕ if necessary, it can be assumed that $\phi_k(i) = 1$ for every k. Define the following matrix-valued function:

$$\begin{aligned} \Psi\colon Q_0 &\longrightarrow \mathrm{End}(\mathbb{C}^{Q_0}) \\ j &\longmapsto \mathrm{diag}(\phi_1(j), \ldots, \phi_m(j)) . \end{aligned}$$

It is clear that Ψ is a solution to the diagonalised fusion problem (3.9). Hence $\Phi = U\Psi U^{-1}$ is a solution to the original fusion problem (3.8). Moreover,

$$\Phi(i) = U\Psi(i)U^{-1} = UIU^{-1} = I$$

as required. □

3.4.3 The $\mathrm{SU}(2)_k$ and $\mathrm{SU}(3)_k$ quivers

In this section, it is shown that Theorem 3.4.2 applies to the truncated $\mathrm{SU}(2)_k$ and $\mathrm{SU}(3)_k$ quivers.

The $\mathrm{SU}(2)_k$ quivers

The $\mathrm{SU}(2)_k$ quivers are precisely the A_k Dynkin quivers. They can be interpreted as a finite segment of the A_∞ quiver encoding the SU(2) fusion rule:

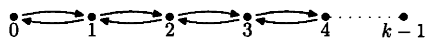


Figure 3-6: The $\mathrm{SU}(2)_k$ quiver.

The fusion rule defined by the $\mathrm{SU}(2)_k$ quiver is the truncated SU(2) fusion rule

$$\Phi(n)A = \Phi(n-1) + \Phi(n+1)$$

where A is the adjacency matrix of the $\mathrm{SU}(2)_k$ quiver and with the convention that $\Phi(-1) = \Phi(k) = 0$.

It has already been shown, by analysing eigenvalues, that the A_k quiver produces a finite solution $\{X_i\}$ to the SL_2 recurrence that satisfies $X_{h-1} = 0$. Notice that this is precisely the statement that the $SU(2)_k$ quivers satisfy the $SU(2)_k$ fusion rule with marked vertex 0, in the sense of Theorem 3.4.2. It is enlightening, however, to give an elementary proof of the sufficient conditions for the theorem to apply. This will better motivate the discussion for the $SU(3)_k$ quivers.

The adjacency matrix of an $SU(2)_k$ quiver is symmetric, hence diagonalisable. It remains to show that each eigenvalue of an $SU(2)_k$ quiver has a corresponding eigenvector that does not vanish on vertex 0. Suppose for contradiction that this is false. Then, in particular, there is an eigenvector ϕ for the $SU(2)_k$ quiver satisfying $\phi(0) = 0$. Denote the corresponding eigenvalue by λ. Then

$$0 = \lambda\phi(0) = \phi(1) .$$

A simple induction now proves that ϕ is identically zero and hence cannot be an eigenvector.

Theorem 3.4.2 can now be applied to yield the result that there is a solution Φ to the $SU(2)_k$ fusion rule satisfying $\Phi(0) = I$.

The $SU(3)_k$ quivers

The $SU(3)_k$ quivers are precisely the Di Francesco-Zuber $\mathcal{A}^{(k+2)}$ quivers. They encode a finite portion of the $SU(3)$ fusion rule:

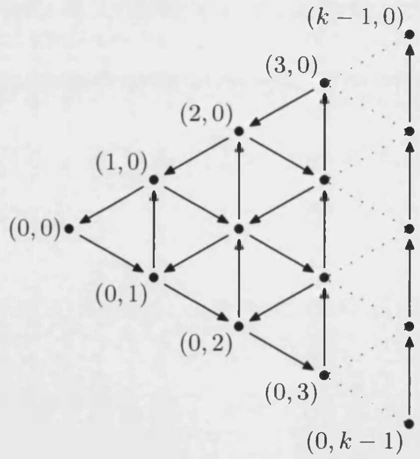


Figure 3-7: The $SU(3)_k$ quiver.

The fusion rule defined by the $\mathrm{SU}(3)_k$ quiver is the truncated SU(3) fusion rule

$$\Phi(i,j)A = \Phi(i+1,j) + \Phi(i-1,j+1) + \Phi(i,j-1)$$

where A is the adjacency matrix of the $\mathrm{SU}(3)_k$ quiver and with the convention that $\Phi(m,n) = 0$ whenever an index is negative or $m+n=k$.

The following simple result, known to Di Francesco and Zuber ([DFZ90a]), permits A and A^T to be treated on an equal footing.

Proposition 3.4.3. *The* $\mathrm{SU}(3)_k$ *quivers have normal adjacency matrices (for the standard inner product).*

Proof. The proof is very visual. Let A denote the adjacency matrix of the $\mathrm{SU}(3)_k$ quiver. The $(i,j)^{\text{th}}$ entry of AA^T corresponds to pairs of arrows of the form

$$i \xrightarrow{a} h(a) = h(b) \xleftarrow{b} j\ .$$

Likewise, the $(i,j)^{\text{th}}$ entry of A^TA corresponds to pairs of arrows of the form

$$i \xleftarrow{a'} t(a') = t(b') \xrightarrow{b'} j;.$$

For $i=j$ the only possibility is that $a=b$ and $a'=b'$, that is, leave and return along the same arrow. However, every vertex of the $\mathrm{SU}(3)_k$ quiver possesses as many incoming arrows as outgoing arrows, whence the diagonal entries of AA^T and A^TA coincide.

It remains to consider $i \neq j$. A pair of arrows of the form $i \to k \leftarrow j$ uniquely determines a rhombus in the $\mathrm{SU}(3)_k$ quiver consisting of two triangles and one internal arrow c common to both. The remaining sides of the rhombus are a pair of arrows of the form $i \leftarrow h(c) \to j$. This defines a one-to-one correspondence between pairs of arrows of the form $i \to k \leftarrow j$ and $i \leftarrow k' \to j$. Consequently the off-diagonal entries of AA^T and A^TA also coincide. □

In particular:

Corollary 3.4.4. *The adjacency matrix A of the* $\mathrm{SU}(3)_k$ *quiver is diagonalisable. The eigenvectors for A and A^T coincide.*

Proposition 3.4.5. *No eigenvector of the* $\mathrm{SU}(3)_k$ *quiver vanishes at vertex* $(0,0)$.

Proof. Suppose otherwise, for contradiction. Then there is an eigenvector ϕ for the $\mathrm{SU}(3)_k$ quiver satisfying $\phi(0,0) = 0$. Let A denote the adjacency matrix of the $\mathrm{SU}(3)_k$

quiver and denote by λ the eigenvalue corresponding to ϕ. Then

$$0 = \lambda\phi(0,0) = \phi(0,1) .$$

Moreover, since ϕ must also be an eigenvector for A^T (that is, it is an eigenvector for the dual quiver, obtained by reversing the direction of each arrow) it also follows that $\phi(1,0) = 0$. It has been shown therefore that $\phi(m,n) = 0$ for every vertex (m,n) satisfying $m+n \leq 1$. Suppose, as an induction hypothesis, that $\phi(m,n) = 0$ for every vertex (m,n) satisfying $m+n \leq p$. Let (m,n) be a vertex satisfying $m+n = p$. Then

$$0 = \lambda\phi(m,n) = \phi(m+1,n) + \phi(m-1,n+1) + \phi(m,n-1) = \phi(m+1,n)$$

using the induction hypothesis and with the usual convention that the value is zero whenever an index is negative. Moreover, ϕ is an eigenvector for A^T and hence $\phi(p+1,0)$ is also zero. It has been shown therefore, that $\phi(m,n) = 0$ for every vertex (m,n) satisfying $m+n = p+1$. By induction, ϕ is identically zero. But this contradicts that ϕ is an eigenvector. □

In particular, the hypotheses of Theorem 3.4.2 are satisfied for the $\mathrm{SU}(3)_k$ quiver. Consequently:

Corollary 3.4.6. *There is a solution Φ to the truncated* $\mathrm{SU}(3)_k$ *fusion rule satisfying $\Phi(0,0) = I$.*

In fact, more is true. Writing out the fusion rule at vertex $(0,0)$ yields $\Phi(1,0) = A$, the adjacency matrix of the $\mathrm{SU}(3)_k$ quiver. Furthermore:

Proposition 3.4.7. *The solution Φ to the truncated* $\mathrm{SU}(3)_k$ *fusion rule constructed in the proof of Theorem 3.4.2 also solves the* dual *fusion rule:*

$$\sum_{j \leftarrow i} \Phi(j) = \Phi(i)A^T .$$

Proof. This is because if ϕ is an eigenvector for A with eigenvalue λ, then it must be an eigenvector for A^T with eigenvalue $\overline{\lambda}$. □

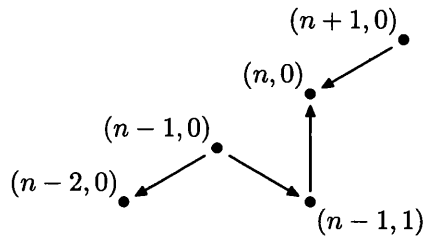
The chapter concludes with a result that is crucial to the analysis in Chapter 7.

Theorem 3.4.8. *Let $\{X_i\}$ be the solution to the* SL_3 *recurrence*

$$X_{i+1} = X_i A - X_{i-1}A^T + X_{i-2} , \quad i \geq 2 ,$$
$$X_0 = I , \quad X_1 = A , \quad X_2 = A^2 - A^T ,$$

where A is the adjacency matrix of the Di Francesco-Zuber $\mathcal{A}^{(h)}$ quiver. Then $X_{h-2} = 0$.

Proof. Corollary 3.4.6 assures that the $\mathcal{A}^{(h)}$ quivers satisfy the $\mathrm{SU}(3)_k$ fusion rule for $k = h - 2$. Let $\Phi(i, j)$ denote the solution to this fusion rule constructed in the proof of Theorem 3.4.2. Consider a small portion of the fusion equations illustrated in the figure below:



By Proposition 3.4.7, Φ is necessarily a solution to the dual fusion problem and thus the fusion rules yield:

$$\begin{aligned}\Phi(n,0)A &= \Phi(n+1,0) + \Phi(n-1,1) \ ,\\ \Phi(n-1,0)A^T &= \Phi(n-1,1) + \Phi(n-2,0) \ .\end{aligned}$$

In particular,

$$\begin{aligned}\Phi(n+1,0) &= \Phi(n,0)A - \Phi(n-1,1)\\ &= \Phi(n,0)A - \Phi(n-1,0)A^T + \Phi(n-2,0) \ .\end{aligned}$$

Let $X_i = \Phi(i, 0)$. Then X_i is the solution to the SL_3 recurrence with initial conditions $X_0 = I$, $X_1 = A$ and $X_2 = A^2 - A^T$. Finally, $X_{h-2} = \Phi(h-2, 0) = 0$, since $h-2 = k$. □

Corollary 3.4.9. *Let $\{X_i\}$ be the solution to the SL_3 recurrence*

$$\begin{gathered}X_{i+1} = X_iA - X_{i-1}A^T + X_{i-2} \ , \quad i \geq 2 \ ,\\ X_0 = I \ , \quad X_1 = A \ , \quad X_2 = A^2 - A^T \ ,\end{gathered}$$

where A is the adjacency matrix of a Di Francesco-Zuber quiver. Then $X_{h-2} = 0$.

Proof. Let Γ be a Di Francesco-Zuber quiver. Then the spectrum of Γ is a subset of the spectrum of $\mathcal{A}^{(h)}$ for some h. Now, since everything is diagonalisable, $X_{h-2} = 0$. □

Chapter 4

Almost Koszul duality

The existence of almost Koszul algebras for the Dynkin quivers is interpreted as an algebraic structure behind the finite solution to the SL_2 recurrence. It is shown that the tadpole quivers also possess almost Koszul algebras.

4.1 Finite solutions from almost Koszul pairs

The concept of *almost Koszul* pair is introduced and is shown to yield finite solutions to the SL_2 recurrence.

4.1.1 Almost Koszul pairs

Fix a semisimple algebra S over $\mathbb{C}$. Let $\Pi_\bullet$ be a bounded graded algebra over S and define p (the HEIGHT of Π) to be the largest integer for which $\Pi_p \neq 0$. Let $\Lambda_\bullet$ be a bounded graded coalgebra over S satisfying $\Lambda_1 \cong \Pi_1$ and define q to be the largest integer for which $\Lambda_q \neq 0$. Suppose that $\Pi_\bullet \otimes_S \Lambda_\bullet$ is a differential complex for the Koszul differential δ. The pair (Π, Λ) is said to be ALMOST KOSZUL if the Koszul differential δ is exact except in degree $(0,0)$ and degree (p,q).

4.1.2 The Dynkin examples

Brenner, Butler and King [BBK02] introduced the idea of almost Koszul duality and proved that the Dynkin quivers support almost Koszul pairs.

Let Q be a Dynkin quiver together with a symplectic form ω. Recall that the pre-projective algebra of (Q, ω) is defined as the quotient of the path algebra $\mathbb{C}Q$ by the ideal $\mathfrak{J}$ of relations of the form

$$\sum_{c_{pi}} \overline{c}_{ip} c_{pi} \ , \qquad i \in Q_0 \ ,$$

where $\overline{c}$ is an element of the dual basis determined by ω (see Section 1.3). The preprojective algebra of a Dynkin quiver is bounded, with height $h-2$, where h is the Coxeter number of the Dynkin quiver.

The TRIVIAL EXTENSION COALGEBRA of the Dynkin quiver (Q, ω) is the graded S-bimodule $\Lambda = S \oplus V \oplus S$ together with the S-linear (coassociative) coproduct Δ defined by

$$\begin{aligned} \Delta(1) &= 1 \otimes 1 \ , \\ \Delta(a_{ij}) &= a \otimes 1 + 1 \otimes a \ , \\ \Delta(e_i) &= e_i \otimes 1 + \sum_{c_{pi}} \overline{c}_{ip} \otimes c_{pi} + 1 \otimes e_i \ . \end{aligned}$$

The trivial extension coalgebra is clearly bounded, with height 2.

Theorem 4.1.1 ([BBK02]). *The preprojective algebra and the trivial extension coalgebra for a Dynkin quiver form an almost Koszul pair.*

A new proof of Theorem 4.1.1 will be seen in the next chapter.

4.1.3 Finite solutions from almost Koszul pairs

Let (Q, ω) be a Dynkin quiver and denote the preprojective algebra by Π and the trivial extension coalgebra by Λ. Then (Π, Λ) is an almost Koszul pair, hence the following collection of sequences is exact:

$$
\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[dr] & & & \\
0 \arrow[dr] & \Pi_0 \otimes \Lambda_2 \arrow[dr] & \Pi_0 \otimes \Lambda_1 \arrow[dr] & & \\
\vdots & \Pi_1 \otimes \Lambda_2 \arrow[dr] & \Pi_1 \otimes \Lambda_1 \arrow[dr] & \Pi_1 \otimes \Lambda_0 \arrow[dr] & \\
0 \arrow[dr] & \vdots & \Pi_2 \otimes \Lambda_1 \arrow[dr] & \Pi_2 \otimes \Lambda_0 \arrow[dr] & 0 \\
0 \arrow[dr] & \Pi_{h-4} \otimes \Lambda_2 \arrow[dr] & \vdots & \Pi_3 \otimes \Lambda_0 \arrow[dr] & 0 \\
 & \Pi_{h-3} \otimes \Lambda_2 \arrow[dr] & \Pi_{h-3} \otimes \Lambda_1 \arrow[dr] & \vdots & 0 \\
 & & \Pi_{h-2} \otimes \Lambda_1 \arrow[dr] & \Pi_{h-2} \otimes \Lambda_0 \arrow[dr] & \vdots \\
 & & & 0 & 0
\end{tikzcd}
$$

Let X_i denote the S-decomposition matrix of Π_i. The S-decomposition matrix of $\Lambda_1 = V$ is the adjacency matrix A of the Dynkin quiver Q. Thus the matrices X_i are the solution to an initial segment of the SL_2 recurrence:

$$
\begin{aligned}
X_{i+1} &= X_i A - X_{i-1} \quad , \quad 1 \le i \le h-2 \, , \\
X_0 &= I \, , \ \ X_1 = A \quad .
\end{aligned}
$$

Permitting the recurrence to continue yields the periodic solution

$$
\begin{aligned}
X_{h-1+i} &= -X_{h-1-i} && \text{for } 0 \le i \le h-1 \, , \\
X_{i+2h} &= X_i && \text{for } i \ge 0 \, .
\end{aligned}
$$

Hence, almost Koszul pairs for the Dynkin quivers give rise to finite solutions of the SL_2 recurrence. Do the tadpole quivers support almost Koszul pairs? Do the $\widehat{\mathfrak{sl}}_3$ rational boundary conformal field theory quivers support almost Koszul pairs for a coalgebra Λ with graded S-bimodule structure $S \oplus V \oplus V^* \oplus S$?

4.2 Almost Koszul pairs for the tadpole quivers

The tadpole quivers are shown to support almost Koszul pairs. A second proof of this emerges from the analysis in the next chapter.

Consider the Dynkin quiver A_{2n}. Denote the group of graph automorphisms on A_{2n} by $G = C_2$ and let τ denote the unique non-trivial graph automorphism. Notice that τ is fixed-point free. The action of G on A_{2n} extends uniquely to an action on the path algebra $\mathbb{C}A_{2n}$ that is compatible with the product. The first claim is that the fixed-point algebra $(\mathbb{C}A_{2n})^G$ is isomorphic to the path algebra $\mathbb{C}T_n$ of the tadpole quiver T_n.

Write $S = (\mathbb{C}A_{2n})_0$. Let V be an S-bimodule and suppose V carries a compatible G-action, that is, for $s, s' \in S$ and for $v \in V$,

$$\tau(s \cdot v \cdot s') = \tau(s) \cdot \tau(v) \cdot \tau(s') \ .$$

Then V^G is an S^G-bimodule. It is clear that $S^G \cong (\mathbb{C}T_n)_0$. Furthermore, $(\mathbb{C}A_{2n})_1^G \cong (\mathbb{C}T_n)_1$ as S^G-bimodules, whence there is an isomorphism of tensor algebras over S^G,

$$T((\mathbb{C}A_{2n})_1^G) \cong \mathbb{C}T_n \ .$$

For the claim, it remains to show that $T((\mathbb{C}A_{2n})_1^G) \cong (\mathbb{C}A_{2n})^G$ as algebras. The universal property of the tensor algebra ensures that there is a unique map of algebras:

$$\begin{tikzcd}
(\mathbb{C}A_{2n})_1^G \arrow[r] \arrow[d] & (\mathbb{C}A_{2n})^G \\
T((\mathbb{C}A_{2n})_1^G) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

This algebra homomorphism is graded. The following result shows that the map is isomorphic on each graded piece:

Lemma 4.2.1. *Let V and W be S-bimodules carrying a compatible G-action. Then there is an isomorphism of S^G-bimodules*

$$\begin{aligned} V^G \otimes_{S^G} W^G &\longrightarrow (V \otimes_S W)^G \\ v \otimes w &\longmapsto v \otimes w \,. \end{aligned}$$

Proof. The inverse can be constructed via the map

$$\begin{aligned} f\colon V \times W &\longrightarrow V^G \otimes_{S^G} W^G \\ (v, w) &\longmapsto 2 \sum_{i=1}^{2n} \pi_0(ve_i) \otimes \pi_0(e_i w) \,, \end{aligned}$$

where π_0 is the projection to the trivial isotypic component and e_i denotes the trivial path at vertex i of the A_{2n} quiver. It is now easy to show that f factorises through $V \otimes_S W$. There is thus a well-defined map

$$(V \otimes_S W)^G \longrightarrow V \otimes_S W \longrightarrow V^G \otimes_{S^G} W^G \,.$$

The proof that this is the inverse map is straightforward, but depends crucially on the fact that the graph automorphism τ is fixed-point free (on vertices). □

It is clear that the map of algebras is of precisely this form. This concludes the proof that $(\mathbb{C}A_{2n})^G$ is isomorphic to $\mathbb{C}T_n$.

Choose a G-equivariant symplectic form ω for A_{2n}. Consider now the ideal $\mathfrak{J}$ in $\mathbb{C}A_{2n}$ generated by relations of the form

$$\sum_{c_{pi}} \overline{c}_{ip} c_{pi} \,, \qquad i \in Q_0 \,.$$

The quotient of $\mathbb{C}A_{2n}$ by $\mathfrak{J}$ is the preprojective algebra Π of A_{2n}. Notice that $\mathfrak{J}$ is closed under the action of G, hence there is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{J}^G \arrow[r] \arrow[d] & (\mathbb{C}A_{2n})^G \arrow[r] \arrow[d] & \Pi^G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{J} \arrow[r] & \mathbb{C}A_{2n} \arrow[r] & \Pi \arrow[r] & 0 \,. \end{tikzcd}$$

Since $(\mathbb{C}A_{2n})^G \cong \mathbb{C}T_n$ and $\mathfrak{J}^G$ is an ideal of $(\mathbb{C}A_{2n})^G$, the fixed part of the preprojective algebra, Π^G, is an algebra with relations on T_n. The product $\overline{\mu}$ on Π^G coincides with the obvious graded map:

$$\Pi_i^G \otimes_{S^G} \Pi_j^G \xrightarrow{\cong} (\Pi_i \otimes_S \Pi_j)^G \longrightarrow \Pi_{i+j}^G \,.$$

To see this, notice first that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{I}_i^G \otimes_{S^G} \mathfrak{I}_j^G \arrow[r, "\cong"] \arrow[d] & (\mathfrak{I}_i \otimes_S \mathfrak{I}_j)^G \arrow[r] \arrow[d] & \mathfrak{I}_{i+j}^G \arrow[d] \\
(\mathbb{C}A_{2n})_i^G \otimes_{S^G} (\mathbb{C}A_{2n})_j^G \arrow[r, "\cong"] \arrow[d] & ((\mathbb{C}A_{2n})_i \otimes_S (\mathbb{C}A_{2n})_j)^G \arrow[r] \arrow[d] & (\mathbb{C}A_{2n})_{i+j}^G \arrow[d] \\
\Pi_i^G \otimes_{S^G} \Pi_j^G \arrow[r, "\cong"] & (\Pi_i \otimes_S \Pi_j)^G \arrow[r] & \Pi_{i+j}^G \,.
\end{tikzcd}$$

Now the product $\Pi_i^G \otimes_{S^G} \Pi_j^G \to \Pi_{i+j}^G$ is by definition: lift an element of $\Pi_i^G \otimes_{S^G} \Pi_j^G$ to $(\mathbb{C}A_{2n})_i^G \otimes_{S^G} (\mathbb{C}A_{2n})_j^G$, take the product to $(\mathbb{C}A_{2n})_{i+j}^G$ (both ways of doing this coincide, so in particular the one in the above diagram can be used) and then project onto the quotient Π_{i+j}^G. Commutativity of the above diagram now ensures that this coincides with the composition of maps in the bottom row, as claimed.

Consider next $\Lambda^G \cong S^G \oplus (\mathbb{C}T_n)_1 \oplus S^G$. It is easily verified that the coproduct on Λ is G-equivariant and induces a coassociative S^G-linear coproduct $\overline{\Delta}$ on Λ^G defined by

$$\Lambda^G \xrightarrow{\Delta^G} (\Lambda \otimes_S \Lambda)^G \xrightarrow{\cong} \Lambda^G \otimes_{S^G} \Lambda^G \,.$$

The remaining assertion is that (Π^G, Λ^G) is an almost Koszul pair. Let δ denote the Koszul differential on $\Pi \otimes_S \Lambda$. Notice that exactness of the differential δ^G on $(\Pi \otimes_S \Lambda)^G$ is immediate, however it is required to show that this differential coincides with the Koszul differential $\overline{\delta}$ on $\Pi^G \otimes_{S^G} \Lambda^G$.

To this end consider the following diagram:

$$\begin{tikzcd}
(\Pi_p \otimes_S \Lambda_q)^G \arrow[rrr, "\delta^G"] \arrow[dr, "(1\otimes\Delta)^G"] \arrow[ddd, "\cong"'] & & & (\Pi_{p+1} \otimes_S \Lambda_{q-1})^G \arrow[ddd, "\cong"] \\
& (\Pi_p \otimes_S \Lambda_1 \otimes_S \Lambda_{q-1})^G \arrow[r, "\cong"] \arrow[d, "\cong"] & (\Pi_p \otimes_S \Pi_1 \otimes_S \Lambda_{q-1})^G \arrow[ur, "(\mu\otimes 1)^G"] \arrow[d, "\cong"'] & \\
& \Pi_p^G \otimes_{S^G} \Lambda_1^G \otimes_{S^G} \Lambda_{q-1}^G \arrow[r, "\cong"] & \Pi_p^G \otimes_{S^G} \Pi_1^G \otimes_{S^G} \Lambda_{q-1}^G \arrow[dr, "\overline{\mu}\otimes 1"'] & \\
\Pi_p^G \otimes_{S^G} \Lambda_q^G \arrow[ur, "1\otimes\overline{\Delta}"'] \arrow[rrr, "\overline{\delta}"] & & & \Pi_{p+1}^G \otimes_{S^G} \Lambda_{q-1}^G
\end{tikzcd} .$$

The top and bottom squares commute by definition of the Koszul differential and it is clear that the central square commutes. The left-hand square commutes by the definition of the coproduct on Λ^G and the right-hand square commutes in light of the discussion about the product on Π. Thus, the differential δ^G coincides with the Koszul differential $\overline{\delta}$ on $\Pi^G \otimes_{S^G} \Lambda^G$ as required.

In the next chapter it is shown how these almost Koszul pairs arise in a universal way.

Chapter 5

The Temperley-Lieb category

For a symplectic quiver, it is shown that Ocneanu's spaces of essential paths can be identified with the graded pieces of the preprojective algebra. Graph Temperley-Lieb representations are introduced as functors on the Temperley-Lieb category supported by the path algebra of a quiver. For generic values of the parameter in the Temperley-Lieb category, it is possible to build a Koszul algebra-coalgebra pair from a graph Temperley-Lieb representation. For singular values of the parameter and given a graph Temperley-Lieb representation factoring through the unique proper tensor ideal in the category, it is possible to construct an almost Koszul algebra-coalgebra pair.

5.1 Essential paths

Essential paths on a symplectic quiver are defined via an action of the Temperley-Lieb algebras on the path algebra of the quiver. The spaces of essential paths are identified with the graded pieces of the preprojective algebra of the symplectic quiver.

5.1.1 A short note on maps

Functions will usually operate on the left and correspondingly will compose right to left. However, operators associated with the Temperley-Lieb category will always be chosen to operate on the right and to compose left to right. This coincides with the use of planar Brauer diagrams to represent maps in the Temperley-Lieb category in Section 5.2 and the choice that they should compose left to right. The notation $g \circ f$ will *always* denote right-to-left composition.

5.1.2 The Temperley-Lieb algebras

Let (Q,ω) be a (finite connected) symplectic quiver with Perron-Frobenius eigenvalue λ and Perron-Frobenius eigenvector $(x_i)_{i\in Q_0}$. Write $V=(\mathbb{C}Q)_1$ and $S=(\mathbb{C}Q)_0$ and let $e_i\in S$ denote the trivial path at the vertex $i\in Q_0$. Recall that the arrows a_{ij} form a basis for V and let $\overline{a}_{ji}$ denote elements of the dual basis determined by ω (see Section 1.3). Define the S-bimodule maps

$$\begin{array}{rrcl} \phi : & V\otimes_S V & \longrightarrow & S \\ & a_{ij}b_{jk} & \longmapsto & (a_{ij}b_{jk})\phi = x_j\, e_i\omega(a_{ij},b_{jk})e_k\ , \end{array}$$

$$\begin{array}{rrcl} \psi : & S & \longrightarrow & V\otimes_S V \\ & e_i & \longmapsto & (e_i)\psi = \frac{1}{x_i}\sum\limits_{c_{pi}\in Q_1}\overline{c}_{ip}c_{pi}\ . \end{array}$$

Lemma 5.1.1. *The maps ϕ and ψ satisfy the relations*

$$\psi\phi = \lambda\ \mathrm{id}_S\ , \tag{5.1}$$

$$(\psi\otimes 1)(1\otimes\phi) = -\mathrm{id}_V\ , \tag{5.2}$$

$$(1\otimes\psi)(\phi\otimes 1) = -\mathrm{id}_V\ . \tag{5.3}$$

Proof. For relation (5.1)

$$\begin{aligned} (e_i)\psi\phi &= \left(\frac{1}{x_i}\sum_{c_{pi}}\overline{c}_{ip}c_{pi}\right)\phi \\ &= \frac{1}{x_i}\sum_{c_{pi}}x_p e_i \\ &= \lambda e_i\ . \end{aligned}$$

Relation (5.2) is satisfied since

$$\begin{aligned} (a_{ij})\left((\psi\otimes 1)(1\otimes\phi)\right) &= \left(\frac{1}{x_i}\sum_{c_{pi}}\overline{c}_{ip}c_{pi}a_{ij}\right)(1\otimes\phi) \\ &= \frac{1}{x_i}\sum_{c_{pi}}\overline{c}_{ip}\, e_p\omega(c_{pi},a_{ij})e_j\, x_i \\ &= -a_{ij}\ . \end{aligned}$$

The remaining equation is entirely analogous. □

Define the Temperley-Lieb operator $U = \phi\psi\colon (\mathbb{C}Q)_2 \to (\mathbb{C}Q)_2$. Fix $n \geq 2$. For $1 \leq i \leq n-1$ define the tensor extension of the Temperley-Lieb operator $U_i\colon (\mathbb{C}Q)_n \to (\mathbb{C}Q)_n$ by

$$U_i = \overbrace{1 \otimes \cdots \otimes 1}^{i-1 \text{ times}} \otimes U \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-i-1 \text{ times}} .$$

Then it is easily shown that the U_i satisfy the defining relations of the Temperley-Lieb algebra $TL_n(\lambda)$:

$$
\begin{aligned}
U_i^2 &= \lambda U_i\,, & 1 \leq i \leq n-1\,, && (5.4)\\
U_iU_{i+1}U_i &= U_i\,, & 1 \leq i \leq n-2\,, && (5.5)\\
U_iU_{i-1}U_i &= U_i\,, & 2 \leq i \leq n-1\,, && (5.6)\\
U_iU_j &= U_jU_i\,, & 1 \leq i,j \leq n-1\,,\ |i-j| \geq 2\,. && (5.7)
\end{aligned}
$$

Remark 5.1.2. *If a symmetric quiver had been chosen in place of a symplectic quiver then formulae* (5.2) *and* (5.3) *contain a plus sign, rather than the minus sign. This does not change the fact that the above construction yields a collection of representations of the Temperley-Lieb algebras. There is some subtlety regarding the choice of symplectic quivers over symmetric quivers in this thesis, which will be detailed shortly.*

Remark 5.1.3. *The Perron-Frobenius eigenvector is chosen here only because all its components are non-zero. Any eigenvector with this property would suffice.*

Definition 5.1.4 (d'après Ocneanu, Coquereaux [Coq02]). *Let $p \in (\mathbb{C}Q)_n$ be a path of length n. Then p is an* ESSENTIAL PATH *of length n if*

$$p \in \bigcap_{i=1}^{n-1} \mathrm{Ker}(U_i)\,.$$

The space of essential paths of length n is denoted $\mathrm{EssPath}_n$.

5.1.3 Preprojective algebras

The preprojective algebra of a symplectic quiver (Q,ω) is the quotient of the path algebra by the ideal of relations of the form

$$\sum_{c_{pi}} \overline{c}_{ip}c_{pi}\,, \qquad i \in Q_0\,,$$

where the sum is taken over all arrows c with head at the fixed vertex i and tail at another vertex p. This ideal coincides precisely with the ideal generated by $\mathrm{Im}(U)$.

Coquereaux shows [CG05] that the spaces of essential paths carry an algebra structure. This result is proved in the language of this thesis, from which it follows that the algebra of essential paths and the preprojective algebra of a symplectic quiver are (often) isomorphic.

Let Q be a non-Dynkin symplectic quiver. Then it is a well-known result (see [BBK02]) that the preprojective algebra of (Q, ω) is Koszul. If Q is a Dynkin quiver then it has been shown [BBK02] that the preprojective algebra of (Q, ω) is almost Koszul.

The remainder of this chapter is devoted to proving the existence of Koszul or almost Koszul algebras for a symplectic quiver. The proof in the Koszul case is not new (cf. [MOV06]), but with an emphasis on exploiting the diagram calculus of the Temperley-Lieb category. For the almost Koszul case, the analysis draws on the work of Goodman and Wenzl [GW03] to produce a more complete picture than seems to appear in the current literature. This proof is central to establishing a connection between almost Koszul algebras and lattice models for the $c < 1$ rational conformal field theories.

5.2 The Temperley-Lieb categories

The Temperley-Lieb categories $\underline{\mathrm{TL}}^+$ and $\underline{\mathrm{TL}}^-$ are introduced.

5.2.1 Planar Brauer diagrams

This section follows the treatment by Goodman and Wenzl [GW03]. Let m and n be non-negative integers of the same parity. A planar Brauer diagram from m to n consists of the following:

1. a closed rectangle R in the plane with two opposite edges designated as *left* and *right*;
2. m marked points on the left edge and n marked points on the right edge;
3. $\frac{m+n}{2}$ smooth curves in R such that the curves are pairwise non-intersecting and such that for each curve γ, the set $\gamma \cap \partial R$ consists of two of the $n + m$ marked points.

The rectangle containing no marked points and no curves is the only planar Brauer diagram from 0 to 0.

Two such diagrams are *equivalent* if they induce the same pairing on the $n+m$ marked points. Each equivalence class of diagrams will still be referred to as a diagram. An example of a planar Brauer diagram is in Figure 5-1 below.

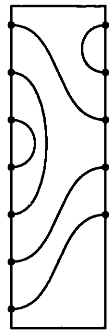


Figure 5-1: A planar Brauer diagram from 7 to 5.

Henceforth, the rectangle will be omitted from the diagram.

5.2.2 Defining the Temperley-Lieb categories

The Temperley-Lieb category $\underline{\mathrm{TL}}^+ = \underline{\mathrm{TL}}^+[\delta]$ is a $\mathbb{C}[\delta]$-linear category whose objects are columns of n dots ($n = 0, 1, 2, 3, \ldots$) denoted $\underline{n}$. A $\mathbb{C}[\delta]$-basis for $\mathrm{Hom}(\underline{m}, \underline{n})$ is the set of planar Brauer diagrams from m to n. Composition of morphisms is defined first on diagrams.

Let A be a planar Brauer diagram from m to n and let B be a planar Brauer diagram from n to p. The composition AB of the diagrams A and B is defined as follows:

1. Juxtapose the rectangles of A and B, identifying the right edge of A (with its n marked points) with the left edge of B (with its n marked points).

2. Remove from the resulting rectangle any closed loops in its interior. The result is a planar Brauer diagram C from m to p.

3. The product AB is $\delta^r C$, where r is the number of loops removed in step 2.

Composition of morphisms is the bilinear extension of the composition of diagrams.

A strand in a Brauer diagram is called a THROUGH-STRING if it connects a point on the left edge of the rectangle to a point on the right edge. It is easily seen that the number of through-strings in a diagram is equal to the smallest n such that the diagram factorises through $\underline{n}$. Consequently, the product in the Temperley-Lieb category is filtered for the filtration $\mathcal{F}_n$ defined by

$$\mathcal{F}_n = \{\text{morphisms } f \ : \ f \text{ factorises through } \underline{n}\} .$$

The Temperley-Lieb category $\underline{\mathrm{TL}}^+$ is a monoidal category with tensor product on objects defined by $\underline{m} \otimes \underline{n} = \underline{m+n}$. The tensor product of two planar Brauer diagrams, A and B, is the planar Brauer diagram obtained by stacking the two diagrams vertically:

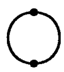
$$A \otimes B = \begin{matrix} A \\ B \end{matrix} .$$

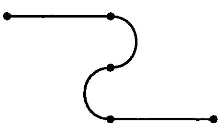
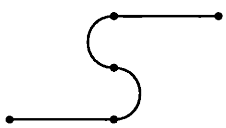
The tensor product can now be extended to arbitrary pairs of morphisms by bilinearity. It is a well-known result, essentially due to Kauffman [Kau87], that the Temperley-Lieb algebra TL_n coincides with the endomorphism algebra $\mathrm{End}(\underline{n})$ in $\underline{\mathrm{TL}}^+$.

The Temperley-Lieb category $\underline{\mathrm{TL}}^+$ is generated under composition and tensor products (see [Kau90] for example) by the creation and annihilation operators

 & 

subject to the relations

 $=$ δ

 $=$ $=$  .

Strictly speaking, the category pertinent to the thesis will not be the category $\underline{\mathrm{TL}}^+$, but the category, $\underline{\mathrm{TL}}^-$, obtained by twisting the product on $\underline{\mathrm{TL}}^+$.

Proposition 5.2.1. *The relations in Figure 5-2 below extend uniquely to an associative product on* $\underline{TL}^+$.

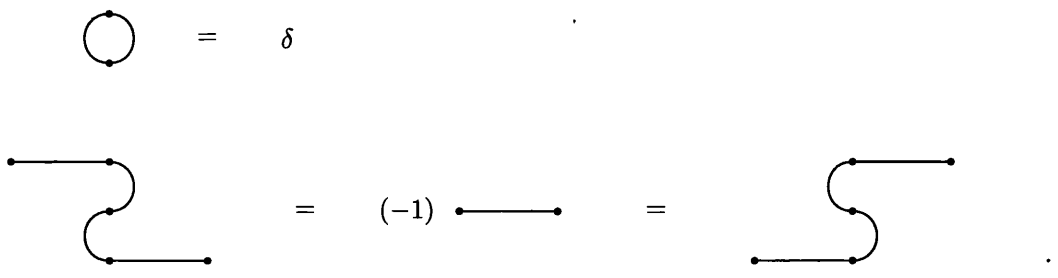


Figure 5-2: The fundamental relations for $\underline{\mathrm{TL}}^{-}$.

Proof. Let A be a diagram from $\underline{m}$ to $\underline{n}$. Denote the dots on the left edge successively by $1, \dots, m$ and the dots on the right edge by $1', \dots, n'$. Consider the set R_A of pairs induced by the diagram A. Define the following function on such pairs:

$$\begin{aligned} \epsilon(a,b) &= (-1)^{\frac{b-a-1}{2}} , \\ \epsilon(a',b') &= (-1)^{\frac{b-a+1}{2}} , \\ \epsilon(a,b') &= 1 . \end{aligned}$$

Extend ϵ to diagrams by setting

$$\epsilon(A) = \prod_{r \in R_A} \epsilon(r) \in \{\pm 1\} .$$

Define the following map on pairs of diagrams

$$(A, B) \longmapsto \epsilon(A)\epsilon(B)\epsilon(AB)AB ,$$

where AB denotes the standard product of A and B in $\underline{\mathrm{TL}}^{+}$, and extend bilinearly to obtain a map

$$\mathrm{Hom}(\underline{m}, \underline{n}) \otimes \mathrm{Hom}(\underline{n}, \underline{p}) \xrightarrow{\mu} \mathrm{Hom}(\underline{m}, \underline{p}) .$$

The identity diagrams $\mathrm{id}_{\underline{n}}$ satisfy $\epsilon(\mathrm{id}_{\underline{n}}) = 1$ and hence remain units for the new product. Moreover, the product is associative since for diagrams A, B, C,

$$\begin{aligned} \mu \circ (\mu \otimes 1)(A \otimes B \otimes C) &= \epsilon(A)\epsilon(B)\epsilon(AB)\mu(AB \otimes C) \\ &= \epsilon(A)\epsilon(B)\epsilon(C)\epsilon(ABC)ABC \\ &= \mu \circ (1 \otimes \mu)(A \otimes B \otimes C) . \end{aligned}$$

In fact, an n-fold product is given by

$$A_1 \otimes \cdots \otimes A_n \longmapsto \epsilon(A_1) \cdots \epsilon(A_n)\epsilon(A_1 \cdots A_n)A_1 \cdots A_n \,. \tag{5.8}$$

It is now easy to see that this is the product generated by the declared relations. The creation and annihilation operators generate $\underline{\mathrm{TL}}^+$ under the standard product, hence (5.8) ensures that they still generate $\underline{\mathrm{TL}}^+$ with the new product. It suffices therefore to notice that the relations coincide with the new product, albeit with δ in place of $-\delta$. $\square$

Definition 5.2.2. *The category $\underline{TL}^- = \underline{TL}^-[\delta]$ coincides with $\underline{TL}^+$ on objects and Hom-sets, but has the twisted product generated by the relations in Figure 5-2.*

A functor between two $\mathbb{C}[\delta]$-linear categories is said to be $\mathbb{C}[\delta]$-ANTILINEAR if it is $\mathbb{C}$-linear and sends δ to $-\delta$. More precisely, the functor commutes with the action of $\mathbb{C}[\delta]$ on Hom-sets, up to a twist by the involution $\delta \mapsto -\delta$.

Proposition 5.2.3. *There is a $\mathbb{C}[\delta]$-antilinear isomorphism between the categories $\underline{TL}^+$ and $\underline{TL}^-$.*

Proof. The functor defined by

$$\begin{array}{rcl} \underline{\mathrm{TL}}^+ & \longrightarrow & \underline{\mathrm{TL}}^- \\ \underline{n} & \longmapsto & \underline{n} \\ A & \longmapsto & \epsilon(A)A \end{array}$$

is a $\mathbb{C}[\delta]$-antilinear isomorphism. $\square$

The prevalence in the literature of $\underline{\mathrm{TL}}^+$ over $\underline{\mathrm{TL}}^-$ can be easily understood: there is no need to keep track of minus signs when working in $\underline{\mathrm{TL}}^+$. It will be important, however, in the course of this thesis, to differentiate between these two categories. In particular, specialising the parameter δ to certain natural values in $\mathbb{R}$ that will subsequently appear, indicates a natural preference for $\underline{\mathrm{TL}}^-$ over $\underline{\mathrm{TL}}^+$.

5.2.3 Specialisations of the Temperley-Lieb categories

The specialisation $\underline{\mathrm{TL}}^\pm(\delta_0)$ of $\underline{\mathrm{TL}}^\pm[\delta]$ is the $\mathbb{C}$-linear category obtained by replacing the parameter δ by $\delta_0 \in \mathbb{C}$ in the construction above. Notice that the Temperley-Lieb algebra $TL_n(\delta_0)$ coincides with $\mathrm{End}(\underline{n})$ in $\underline{\mathrm{TL}}^\pm(\delta_0)$. The $\mathbb{C}[\delta]$-antilinear isomorphism between $\underline{\mathrm{TL}}^+$ and $\underline{\mathrm{TL}}^-$ induces a $\mathbb{C}$-linear isomorphism between the specialisations $\underline{\mathrm{TL}}^+(\delta_0)$ and $\underline{\mathrm{TL}}^-(-\delta_0)$.

5.3 The Yoneda embedding

The analysis of the specialisations $\underline{\mathrm{TL}}^+(\delta_0)$ in the next section will depend on being able to formally adjoin two constructions: the image of an idempotent and the tensor product of an object by a vector space. The most convenient way to describe these constructions is to exhibit an isomorphism between $\underline{\mathrm{TL}}^+(\delta_0)$ and a full subcategory of the category of contravariant functors from $\underline{\mathrm{TL}}^+(\delta_0)$ to $\underline{\mathrm{Vect}}$, in which the additional structure can be exploited. The material in this section is standard (see [Fre64]).

5.3.1 Defining the Yoneda embedding

Let $\mathcal{C}$ be a $\mathbb{C}$-linear category and let $\mathfrak{Fun}^\circ(\mathcal{C},\underline{\mathrm{Vect}})$ denote the category of contravariant functors from $\mathcal{C}$ to the category of $\mathbb{C}$-vector spaces. There is an embedding of $\mathcal{C}$ into $\mathfrak{Fun}^\circ(\mathcal{C},\underline{\mathrm{Vect}})$ via the Yoneda functor

$$\begin{aligned} \yen(\bullet)\colon \mathcal{C} &\longrightarrow \mathfrak{Fun}^\circ(\mathcal{C},\underline{\mathrm{Vect}}) \\ A &\longmapsto \underline{A} = \mathrm{Hom}_{\mathcal{C}}(-,A) \\ f &\longmapsto f_* \,. \end{aligned}$$

The following proposition proves that the Yoneda functor is an embedding.

Proposition 5.3.1. *The Yoneda functor is fully faithful.*

Proof. Let $\phi,\psi \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ satisfying $\phi_* = \psi_*$. Then

$$\phi = \phi_*(\mathrm{id}_X) = \psi_*(\mathrm{id}_X) = \psi \,,$$

whence the Yoneda functor is faithful. Next, consider the natural transformation

$$\chi_\bullet\colon\ \yen(A) \longrightarrow \yen(B)\,.$$

In particular, there is a map

$$\chi_A\colon\ \mathrm{Hom}(A,A) \longrightarrow \mathrm{Hom}(A,B)\,.$$

Define $\chi_A(\mathrm{id}_A) = \psi \in \mathrm{Hom}(A,B)$. Let $Z \in \mathcal{C}$ and fix a morphism $f \in \mathrm{Hom}(Z,A)$. Then the naturality of χ ensures that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r, "\chi_A"] \arrow[d, "f^*"'] & \mathrm{Hom}(A,B) \arrow[d, "f^*"] \\
\mathrm{Hom}(Z,A) \arrow[r, "\chi_Z"'] & \mathrm{Hom}(Z,B)
\end{tikzcd} \cdot$$

In particular,

$$\chi_Z(f) = \chi_Z \circ f^*(\mathrm{id}_A) = f^* \circ \chi_A(\mathrm{id}_A) = f^*(\psi) = \psi \circ f = \psi_*(f)\ .$$

Hence $\chi = \psi_*$ and the Yoneda functor is full. $\square$

Proposition 5.3.2. *Suppose that $\mathcal{C}$ is a small category. Then the Yoneda functor defines an isomorphism of categories $\mathcal{C} \longrightarrow \yen(\mathcal{C})$.*

Proof. The category $\mathcal{C}$ is small, hence there are set-theoretic inverse maps on $\yen(\mathcal{C})$ to the maps induced by the Yoneda functor on objects and morphisms in $\mathcal{C}$. It remains to show that these maps define a functor. Let $\Phi\colon \yen(\mathcal{C}) \to \mathcal{C}$ denote the set-theoretic inverse map to $\yen$ on morphisms. Then for composable morphisms $f, g \in \yen(\mathcal{C})$,

$$\Phi(fg) = \Phi(\yen(\Phi(f))\,\yen(\Phi(g))) = \Phi(\yen(\Phi(f)\Phi(g))) = \Phi(f)\Phi(g)\ .$$

Thus, the set-theoretic inverses to $\yen$ define a functor. $\square$

Proposition 5.3.3. *The functor category $\mathfrak{Fun}^\circ(\mathcal{C}, \underline{\mathrm{Vect}})$ is an abelian category.*

Proof. The structure of an abelian category, including kernels and cokernels, is inherited pointwise from $\underline{\mathrm{Vect}}$. $\square$

The remainder of this section serves to show how the Yoneda embedding can be exploited to enrich the structure of a small $\mathbb{C}$-linear category.

5.3.2 Idempotent completion

Let $\mathcal{C}$ be a small $\mathbb{C}$-linear category and suppose that $e \in \mathrm{End}_{\mathfrak{Fun}^\circ(\mathcal{C}, \underline{\mathrm{Vect}})}(Z)$ is an idempotent. Then the following functor is an image object for e in the abelian category

$\mathfrak{Fun}^\circ(\mathcal{C}, \underline{\mathrm{Vect}})$:

$$\begin{array}{rcl} Z^e\colon \mathcal{C} & \longrightarrow & \underline{\mathrm{Vect}} \\ A & \longmapsto & \mathrm{Im}(e_A) \cong \{v \in Z(A) \ : \ e_A(v) = v\} \\ (A \xrightarrow{f} B) & \longmapsto & Z(f)_{|\mathrm{Im}(e_B)}\colon Z^e(B) \to Z^e(A) \ . \end{array}$$

Notice that $Z^e(f)$ is well-defined by the naturality of e, since for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ the following square commutes:

$$\begin{tikzcd} Z(B) \arrow[r, "e_B"] \arrow[d, "Z(f)"'] & Z(B) \arrow[d, "Z(f)"] \\ Z(A) \arrow[r, "e_A"'] & Z(A) \end{tikzcd} \quad .$$

The functor Z^e also comes equipped with a monic natural map $Z^e \xrightarrow{\mathrm{im}(e)} Z$. The morphism e factorises uniquely through $\mathrm{im}(e)$ as

$$Z \xrightarrow{\mathrm{coim}(e)} Z^e \xrightarrow{\mathrm{im}(e)} Z \ .$$

The following lemma will be important:

Lemma 5.3.4.

$$\mathrm{coim}(e) \circ \mathrm{im}(e) = \mathrm{id}_{Z^e} \ .$$

Proof. Let $v \in Z^e(A) = \mathrm{Im}(e_A)$. Then $(\mathrm{im}(e))_A(v)$ is invariant under e since e is idempotent, whence the result. □

Proposition 5.3.5. *Let $e \in \mathrm{End}(Z)$ be an idempotent. The following isomorphisms are functorial in W:*

$$\mathrm{Hom}(W, Z^e) \cong e_* \mathrm{Hom}(W, Z) = \{\chi \in \mathrm{Hom}(W, Z) \ : \ e \circ \chi = \chi\} \ , \tag{5.9}$$

$$\mathrm{Hom}(Z^e, W) \cong e^* \mathrm{Hom}(Z, W) = \{\chi \in \mathrm{Hom}(Z, W) \ : \ \chi \circ e = \chi\} \ . \tag{5.10}$$

Proof. For (5.9), let $\eta \in \mathrm{Hom}(W, Z^e)$. Then $\mathrm{im}(e) \circ \eta \in \mathrm{Hom}(W, Z)$ and is invariant under left composition with e since

$$e \circ \mathrm{im}(e) \circ \eta = \mathrm{im}(e) \circ \mathrm{coim}(e) \circ \mathrm{im}(e) \circ \eta = \mathrm{im}(e) \circ \eta \ .$$

Hence,

$$(\mathrm{im}(e))_*\colon \mathrm{Hom}(W, Z^e) \longrightarrow e_* \mathrm{Hom}(W, Z)$$

and functoriality with respect to W is immediate. For the inverse map, consider

$$(\mathrm{coim}(e))_*\colon e_* \mathrm{Hom}(W, Z) \longrightarrow \mathrm{Hom}(W, Z^e)\ .$$

Then

$$(\mathrm{coim}(e))_* \circ (\mathrm{im}(e))_* = (\mathrm{coim}(e) \circ \mathrm{im}(e))_* = (\mathrm{id}_{Z^e})_* = \mathrm{id}_{\mathrm{Hom}(W,Z^e)}$$

and

$$(\mathrm{im}(e))_* \circ (\mathrm{coim}(e))_* = (\mathrm{im}(e) \circ \mathrm{coim}(e))_* = e_* = \mathrm{id}_{e_* \mathrm{Hom}(W,Z)}\ .$$

This concludes the proof of (5.9). The proof of (5.10) is entirely analogous. □

In the following proposition and henceforth, composition of morphisms denoted by juxtaposition will be from left to right. The notation $g \circ f$ will continue to denote right-to-left composition.

Proposition 5.3.6. *Let $e \in \mathrm{End}(W)$ and $f \in \mathrm{End}(Z)$ be idempotents. Then*

$$\mathrm{Hom}(W^e, Z^f) \cong e\,\mathrm{Hom}(W, Z) f\ .$$

Proof. The isomorphism is given by

$$\begin{array}{rcl} (\mathrm{coim}\, e)^* \circ (\mathrm{im}\, f)_*\colon \mathrm{Hom}(W^e, Z^f) & \longrightarrow & e\,\mathrm{Hom}(W, Z) f \\ \phi & \longmapsto & (\mathrm{coim}\, e)\phi(\mathrm{im}\, f) \end{array}$$

and the analysis proceeds identically to that for Proposition 5.3.5. □

5.3.3 Tensoring by a vector space

Let $\mathcal{C}$ be a small $\mathbb{C}$-linear category. Then for $A \in \mathcal{C}$ and $V \in \underline{\mathrm{Vect}}$, the category $\mathfrak{Fun}^\circ(\mathcal{C}, \underline{\mathrm{Vect}})$ contains an object that will be interpreted as the tensor product of A with the vector space V.

Define the functor

$$\begin{array}{rcl} \underline{\mathrm{A}} \otimes \mathrm{V}\colon \mathcal{C} & \longrightarrow & \underline{\mathrm{Vect}} \\ Z & \longmapsto & \mathrm{Hom}(Z, A) \otimes V \\ f & \longmapsto & f^* \otimes 1\ . \end{array}$$

Proposition 5.3.7. *The following isomorphisms are functorial in B:*

1. $\mathrm{Hom}(\underline{\mathrm{B}}, \underline{\mathrm{A}} \otimes \mathrm{V}) \cong \mathrm{Hom}(B, A) \otimes V$,
2. $\mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, \underline{\mathrm{B}}) \cong \mathrm{Hom}(V, \mathrm{Hom}(A, B))$.

Proof. For (1), it is necessary to produce a natural isomorphism

$$\chi\colon \mathrm{Hom}(¥(\bullet), \underline{\mathrm{A}} \otimes \mathrm{V}) \longrightarrow \mathrm{Hom}(\bullet, A) \otimes V\,.$$

Fix $B \in \mathcal{C}$ and define

$$\begin{aligned} \chi_B\colon \mathrm{Hom}(\underline{B}, \underline{\mathrm{A}} \otimes \mathrm{V}) &\longrightarrow \mathrm{Hom}(B, A) \otimes V \\ \eta &\longmapsto \eta_B(\mathrm{id}_B)\,. \end{aligned}$$

Write $\eta_B = \sum \phi_u \otimes u \in \mathrm{Hom}(B, A) \otimes V$. Let $f \in \mathrm{Hom}(Z, B)$ and consider the following diagram, which is commutative due to the naturality of η:

$$\begin{tikzcd} \mathrm{Hom}(B,B) \arrow[r, "\eta_B"] \arrow[d, "f^*"'] & \mathrm{Hom}(B,A) \otimes V \arrow[d, "f^* \otimes 1"] \\ \mathrm{Hom}(Z,B) \arrow[r, "\eta_Z"'] & \mathrm{Hom}(Z,A) \otimes V \end{tikzcd}\ .$$

In particular,

$$\eta_Z(f) = \eta_Z \circ f^*(\mathrm{id}_B) = (f^* \otimes 1) \circ \eta_B(\mathrm{id}_B) = (f^* \otimes 1)(\textstyle\sum \phi_u \otimes u) = \sum f\phi_u \otimes u\,. \quad (5.11)$$

First, it is shown that χ is natural. Let $\alpha \in \mathrm{Hom}(C, B)$ and consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}(\underline{B}, \underline{A} \otimes V) \arrow[r, "\chi_B"] \arrow[d, "(\alpha_*)^*"'] & \mathrm{Hom}(B,A) \otimes V \arrow[d, "\alpha^* \otimes 1"] \\ \mathrm{Hom}(\underline{C}, \underline{A} \otimes V) \arrow[r, "\chi_C"'] & \mathrm{Hom}(C,A) \otimes V \end{tikzcd}\ .$$

Then for $\eta \in \operatorname{Hom}(\underline{B}, \underline{\mathrm{A}} \otimes \mathrm{V})$,

$$\begin{aligned}
(\alpha^* \otimes 1) \circ \chi_B(\eta) &= (\alpha^* \otimes 1)(\eta_B(\mathrm{id}_B)) \\
&= (\alpha^* \otimes 1) \circ \eta_B(\mathrm{id}_B) \\
&= \eta_C \circ \alpha^*(\mathrm{id}_B) \\
&= \eta_C(\alpha) \\
&= \eta_C \circ \alpha_*(\mathrm{id}_C) \\
&= (\eta \circ \alpha_*)_C(\mathrm{id}_C) \\
&= \chi_C(\eta \circ \alpha_*) \\
&= \chi_C \circ (\alpha_*)^*(\eta)
\end{aligned}$$

as required.

To show that χ is an isomorphism, define an inverse map to χ_B,

$$\lambda_B \colon \operatorname{Hom}(B, A) \otimes V \longrightarrow \operatorname{Hom}(\underline{B}, \underline{\mathrm{A}} \otimes \mathrm{V}) \ ,$$

as follows: for $\xi = \sum \phi_u \otimes u \in \operatorname{Hom}(B, A) \otimes V$ define $\lambda_B(\xi)$ via

$$\begin{aligned}
(\lambda_B(\xi))_Z \colon \operatorname{Hom}(Z, B) &\longrightarrow \operatorname{Hom}(Z, A) \otimes V \\
f &\longmapsto \sum f\phi_u \otimes u \ .
\end{aligned}$$

Then $\lambda_B(\xi)$ is natural. To see this let $g \in \operatorname{Hom}(W, Z)$ and consider the following diagram:

$$\begin{tikzcd}
\operatorname{Hom}(Z,B) \arrow[r, "(\lambda_B(\xi))_Z"] \arrow[d, "g^*"'] & \operatorname{Hom}(Z,A) \otimes V \arrow[d, "g^* \otimes 1"] \\
\operatorname{Hom}(W,B) \arrow[r, "(\lambda_B(\xi))_W"'] & \operatorname{Hom}(W,A) \otimes V
\end{tikzcd} \cdot$$

Then for $f \in \operatorname{Hom}(Z, B)$,

$$(\lambda_B(\xi))_W \circ g^*(f) = (\lambda_B(\xi))_W(gf) = \sum gf\phi_u \otimes u = (g^* \otimes 1) \circ (\lambda_B(\xi))_Z(f)$$

as required.

It still remains to show that λ_B is inverse to χ_B. Let $\sum \phi_u \otimes u \in \mathrm{Hom}(B,A) \otimes V$. Then

$$\begin{aligned}
\chi_B \circ \lambda_B \left(\sum \phi_u \otimes u\right) &= \chi_B \left(\lambda_B \left(\sum \phi_u \otimes u\right)\right) \\
&= \left(\lambda_B \left(\sum \phi_u \otimes u\right)\right)_B (\mathrm{id}_B) \\
&= \sum \phi_u \otimes u
\end{aligned}$$

and for $\eta \in \mathrm{Hom}(\underline{B}, \underline{\mathrm{A}} \otimes \mathrm{V})$ and $f \in \mathrm{Hom}(Z,B)$,

$$\begin{aligned}
((\lambda_B \circ \chi_B)(\eta))_Z(f) &= (\lambda_B(\eta_B(\mathrm{id}_B)))_Z(f) \\
&= \left(\lambda_B \left(\sum \phi_u \otimes u\right)\right)_Z (f) \\
&= \sum f\phi_u \otimes u \\
&= \eta_Z(f)
\end{aligned}$$

by (5.11). This concludes the proof of (1).

For (2), it is necessary to define a natural isomorphism

$$\psi \colon \mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, ¥(\bullet)) \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(A, \bullet)) .$$

Fix $B \in C$ and let $\eta \in \mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, \underline{B})$. Then

$$\eta_A \colon \mathrm{Hom}(A,A) \otimes V \longrightarrow \mathrm{Hom}(A,B) .$$

Hence η defines a map

$$\begin{aligned}
\psi_B(\eta) \colon V &\longrightarrow \mathrm{Hom}(A,B) \\
v &\longmapsto \eta_A(\mathrm{id}_A \otimes v) .
\end{aligned}$$

First it is shown that ψ is natural. Let $\alpha \in \mathrm{Hom}(B,C)$ and consider the following diagram:

$$
\begin{tikzcd}
\mathrm{Hom}(\underline{A} \otimes V, \underline{B}) \arrow[r, "\psi_B"] \arrow[d, "(\alpha_*)_*"'] & \mathrm{Hom}(V, \mathrm{Hom}(A, B)) \arrow[d, "(\alpha_*)_*"] \\
\mathrm{Hom}(\underline{A} \otimes V, \underline{C}) \arrow[r, "\psi_C"'] & \mathrm{Hom}(V, \mathrm{Hom}(A, C)) \cdot
\end{tikzcd}
$$

Then for $\eta \in \mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, \underline{B})$ and for $v \in V$,

$$
\begin{aligned}
((\alpha_*)_* \circ \psi_B(\eta))(v) &= ((\alpha_*)_*(\psi_B(\eta)))(v) \\
&= \alpha_*((\psi_B(\eta))(v)) \\
&= \alpha_*(\eta_A(\mathrm{id}_A \otimes v)) \\
&= \alpha_* \circ \eta_A(\mathrm{id}_A \otimes v) \\
&= (\psi_C(\alpha_* \circ \eta))(v) \\
&= ((\psi_C \circ (\alpha_*)_*)(\eta))(v) ,
\end{aligned}
$$

whence the naturality of ψ.

To show that ψ is an isomorphism define an inverse map,

$$\phi_B \colon \mathrm{Hom}(V, \mathrm{Hom}(A, B)) \longrightarrow \mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, \underline{B}) ,$$

as follows: let $f \in \mathrm{Hom}(V, \mathrm{Hom}(A, B))$ and for $Z \in \mathcal{C}$ let

$$\sum \beta_u \otimes u \in \underline{\mathrm{A}} \otimes \mathrm{V}(Z) = \mathrm{Hom}(Z, A) \otimes V .$$

Then $\phi_B(f)$ is the map defined by

$$(\phi_B(f))_Z \left(\sum \beta_u \otimes u\right) = \sum (f(u))_* \beta_u = \sum \beta_u f(u) .$$

To show that $\phi_B(f)$ is natural, let $\gamma \in \mathrm{Hom}(W, Z)$ and consider the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Hom}(Z,A)\otimes V \arrow[r, "(\phi_B(f))_Z"] \arrow[d, "\gamma^*\otimes 1"'] & \mathrm{Hom}(Z,B) \arrow[d, "\gamma^*"] \\
\mathrm{Hom}(W,A)\otimes V \arrow[r, "(\phi_B(f))_W"'] & \mathrm{Hom}(W,B)
\end{tikzcd}\quad .$$

Then

$$\begin{aligned}
\gamma^* \circ (\phi_B(f))_Z(\beta_u \otimes u) &= \gamma^*(\beta_u f(u)) \\
&= \gamma\beta_u f(u) \\
&= (\phi_B(f))_W(\gamma\beta_u \otimes u) \\
&= (\phi_B(f))_W \circ (\gamma^* \otimes 1)(\beta_u \otimes u) ,
\end{aligned}$$

whence naturality.

It still remains to show that ϕ_B is inverse to ψ_B. Let $\eta \in \mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, \underline{B})$ and fix $Z \in \mathcal{C}$. Then for $u \in V$ and $\beta_u \in \mathrm{Hom}(Z, A)$,

$$\begin{aligned}
((\phi_B \circ \psi_B)(\eta))_Z(\beta_u \otimes u) &= (\phi_B(\psi_B(\eta)))_Z(\beta_u \otimes u) \\
&= \beta_u(\psi_B(\eta))(u) \\
&= \beta_u\eta_A(\mathrm{id}_A \otimes u) \\
&= \beta_u^* \circ \eta_A(\mathrm{id}_A \otimes u) \\
&= \eta_Z \circ (\beta_u^* \otimes 1)(\mathrm{id}_A \otimes u) \qquad (5.12) \\
&= \eta_Z(\beta_u \otimes u) ,
\end{aligned}$$

where (5.12) results from the naturality of η. For the remaining identity, let $f \in \mathrm{Hom}(V, \mathrm{Hom}(A, B))$ and fix $v \in V$. Then

$$\begin{aligned}
((\psi_B \circ \phi_B)(f))(v) &= (\psi_B(\phi_B(f)))(v) \\
&= (\phi_B(f))_A(\mathrm{id}_A \otimes v) \\
&= f(v) .
\end{aligned}$$

This concludes the proof of (2). □

5.4 Analysis of the Temperley-Lieb categories

5.4.1 Quantum integers

As a preliminary to analysing the Temperley-Lieb categories it is prudent to recall some results about quantum integers. Fix $q \in \mathbb{C} \setminus \{0\}$ and define the quantum integers $[n]_q$ as follows:

- set $[0]_q = 0$ and $[1]_q = 1$;
- define $[2]_q = q + q^{-1}$;
- define the quantum integers inductively via

$$[2]_q[n]_q = [n-1]_q + [n+1]_q \, .$$

Notice that $[2]_q = [2]_{q^{-1}}$ hence $[n]_q = [n]_{q^{-1}}$. Furthermore, it is clear that $[-n]_q = -[n]_q$ and that for $q = 1$ the quantum integers coincide with the classical integers. For $n > 1$, it is straightforward to show that

$$[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{3-n} + q^{1-n}$$

and provided $q \neq \pm 1$, the quantum integer $[n]_q$ has an expression as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \, .$$

One important difference between classical and quantum integers is that it is possible that $[n]_q = 0$. In fact, for $q \neq \pm 1$,

$$\begin{aligned} [n]_q = 0 \quad &\Leftrightarrow \quad q^n - q^{-n} = 0 \\ &\Leftrightarrow \quad q^{2n} = 1 \, . \end{aligned}$$

The analysis of the Temperley-Lieb category in the following section will depend crucially on the invertibility (or otherwise) of each quantum integer $[n]_q$. In light of this, the parameter q will be said to be GENERIC if $[n]_q \neq 0$ for every $n > 0$. The parameter q is said to be SINGULAR if it is not generic. For singular q, the smallest positive integer h such that $[h]_q = 0$ will play an important role and will be called the COXETER NUMBER.

The following example makes the connection between the h above and the Coxeter number of a Dynkin diagram. Fix a Dynkin diagram Q with Coxeter number h. Define $q = \exp \frac{i\pi}{h}$. Then the Perron-Frobenius eigenvalue of Q is $[2]_q = 2\cos\frac{\pi}{h}$ and h is indeed the smallest positive integer such that $[h]_q = 0$. In fact, there are pleasing expressions for the components of the Perron-Frobenius eigenvector in terms of quantum integers (see Figure 5-3).

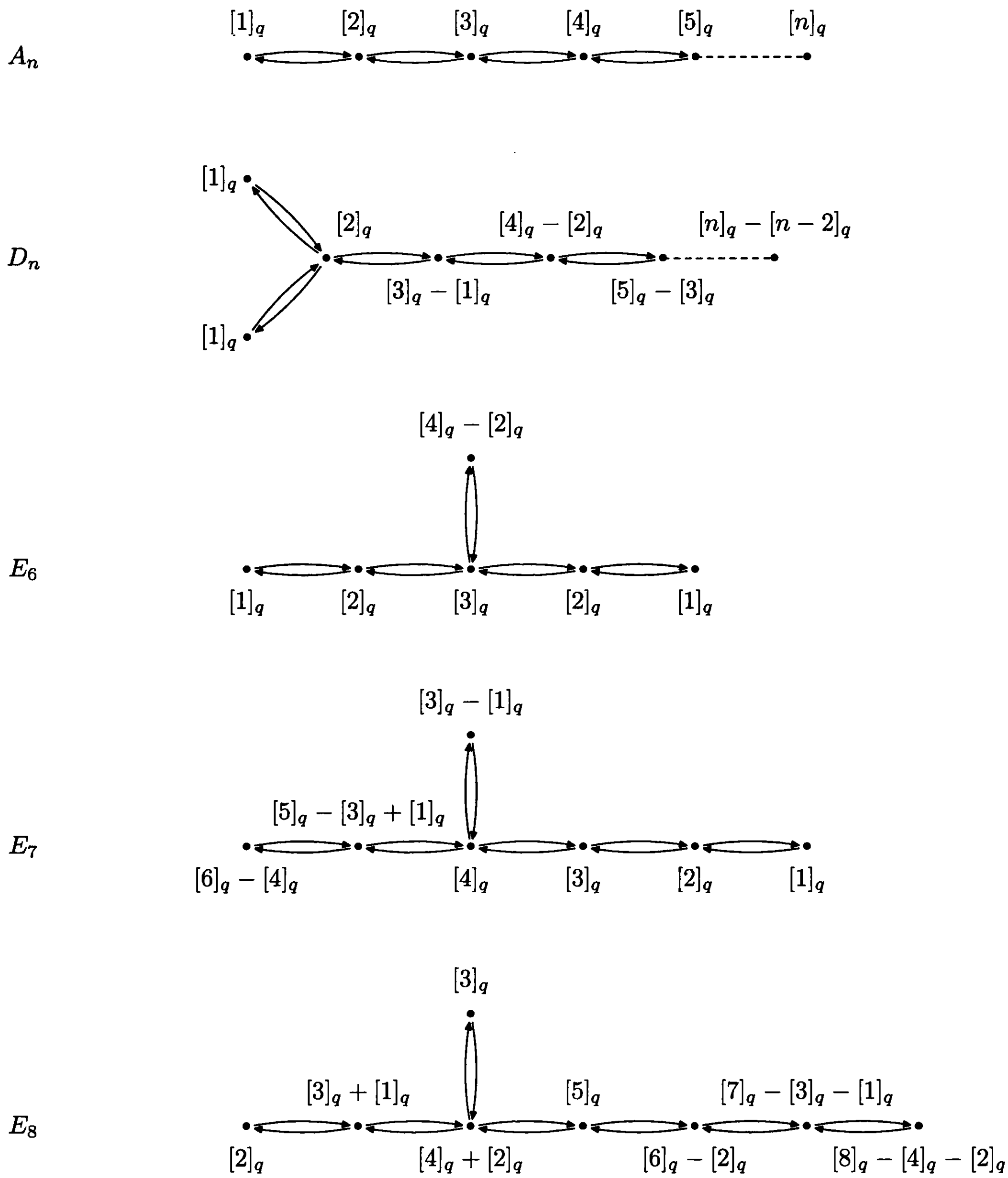


Figure 5-3: Perron-Frobenius eigenvectors for the Dynkin quivers.

In fact, every eigenvalue of a Dynkin quiver is of the form $[2]_q = 2\cos\frac{\pi k}{h}$ for k a Coxeter exponent and $q = \exp\frac{k\pi i}{h}$. For the majority of these q, the above vectors are also eigenvectors with eigenvalue $[2]_q$. This is always the case for the A_n quivers. (The problem, when it occurs, is with the vertex next to the extending vertex in the affine diagram.)

Notice that the components of the eigenvectors for A_n satisfy the scalar SL_2 recurrence defined by the eigenvalue $[2]_q = 2\cos\frac{\pi k}{h}$ with $1 \leq k \leq h-1$ a Coxeter exponent and $q = \exp\frac{k\pi i}{h}$. Returning to the proof of Proposition 3.3.4, it is immediate that, for such q,

$$[n]_q = \frac{\sin\frac{\pi n k}{h}}{\sin\frac{\pi k}{h}} .$$

5.4.2 Analysis of the Temperley-Lieb categories

Fix $\delta_0 \in \mathbb{C}$ and denote $\underline{\mathrm{TL}}^+(\delta_0)$ by $\underline{\mathrm{TL}}$. Write $\delta_0 = q + q^{-1} = [2]_q$ for some $q \in \mathbb{C}\setminus\{0\}$. It is shown that for generic q, every object in $¥(\underline{\mathrm{TL}})$ is semisimple in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$. The analysis begins by identifying a distinguished collection of indecomposables in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ and follows broadly the strategy pursued in [Shi04].

Definition 5.4.1. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. An object X in $\mathcal{C}$ with* $\mathrm{End}(X) \cong \mathbb{C}$ *is said to be a* BRICK.

It is easily shown that in an additive category, a brick is indecomposable.

Let ϕ be a planar Brauer diagram from $\underline{m}$ to $\underline{n}$. Define ϕ° to be the diagram from $\underline{n}$ to $\underline{m}$ obtained by reflecting ϕ in a line parallel to the left and right edges of the rectangle. This operation extends linearly to an automorphism of the Temperley-Lieb category, identifying $\underline{\mathrm{TL}}$ with its opposite category $\underline{\mathrm{TL}}^\circ$.

The monoidal structure on $\underline{\mathrm{TL}}$ induces a monoidal structure on $¥(\underline{\mathrm{TL}})$ by declaring that $¥(\underline{m}) \otimes ¥(\underline{n}) = ¥(\underline{m+n})$ on objects and that on morphisms $\phi_* \otimes \psi_* = (\phi\otimes\psi)_*$. Moreover, this monoidal structure can be extended to the image of idempotents in $¥(\underline{\mathrm{TL}})$ in the following way. Let e be an idempotent in $\mathrm{End}(\underline{m})$ and let f be an idempotent in $\mathrm{End}(\underline{n})$. Define X to be the image object of e_* and Z to be the image object of f_*. Define the tensor product of X and Z to be $X \otimes Z = \mathrm{Im}((e\otimes f)_*)$. Notice that $e \otimes f$ is an idempotent in $\mathrm{End}(\underline{m+n})$ hence the following identities hold:

$$\begin{aligned} \mathrm{Hom}(X\otimes Z, ¥(\underline{p})) &\cong (e\otimes f)^* \,\mathrm{Hom}(\underline{m+n}, \underline{p}) , \\ \mathrm{Hom}(¥(\underline{p}), X\otimes Z) &\cong (e\otimes f)_* \,\mathrm{Hom}(\underline{p}, \underline{m+n}) . \end{aligned}$$

Recall that there is a filtration on morphisms in $\underline{TL}$ and define

$$\mathcal{F}_p(\underline{n}) = \mathcal{F}_p \cap \mathrm{End}(\underline{n}) = \{f \in \mathrm{End}(\underline{n}) \ : \ f \text{ factors through } \underline{p}\} \ .$$

The following result is crucial to subsequent analysis of the Temperley-Lieb category:

Lemma 5.4.2. *Let $\gamma \neq \mathrm{id}_{\underline{n}}$ be a diagram in $\mathrm{End}(\underline{n})$. Then $\gamma \in \mathcal{F}_{n-2}(\underline{n})$. In particular,*

$$\frac{\mathcal{F}_n(\underline{n})}{\mathcal{F}_{n-1}(\underline{n})} \cong \mathbb{C} \ .$$

Proof. It is clear that $\mathrm{id}_{\underline{n}}$ is not an element of $\mathcal{F}_{n-1}(\underline{n})$. Let $\gamma \neq \mathrm{id}_{\underline{n}}$ be a diagram in $\mathrm{End}(\underline{n})$. Then γ must induce at least one pairing of consecutive dots on the left edge of its rectangle and one pairing of consecutive dots on the right edge. Hence γ can have at most $n-2$ through-strings. For the second part of the lemma it suffices to notice that $\mathcal{F}_{n-1}(\underline{n}) = \mathcal{F}_{n-2}(\underline{n})$ by parity and that $\mathrm{id}_{\underline{n}}$ is the only diagram with a non-zero image in $\frac{\mathcal{F}_n(\underline{n})}{\mathcal{F}_{n-1}(\underline{n})}$. □

It will be convenient to write $\delta_0 = q + q^{-1} = [2]_q$ for some $q \in \mathbb{C} \setminus \{0\}$. The following theorem (cf. [Shi04]) identifies a distinguished collection of bricks in $\mathfrak{Fun}^\circ(\underline{TL}, \underline{Vect})$ and recovers the well-known Jones-Wenzl idempotents ([Wen87, Wen88]).

Theorem 5.4.3. *Suppose $[k]_q \neq 0$ for all $1 \leq k \leq n$. Then there is a collection of $n+1$ bricks $\{X_k \ : \ k = 0, \ldots, n\}$ in $\mathfrak{Fun}^\circ(\underline{TL}, \underline{Vect})$ satisfying the fusion rule*

$$\begin{aligned} X_k \otimes X_1 &\cong X_{k-1} \oplus X_{k+1} \qquad \forall 1 \leq k \leq n-1 \ , \\ X_0 \otimes X_1 &= X_1 \ . \end{aligned}$$

Moreover, the X_k form a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{TL}, \underline{Vect})$ generated by direct summands of $\yen(\underline{0}), \yen(\underline{1}), \yen(\underline{2}), \ldots, \yen(\underline{n})$.

The additive subcategory generated by direct summands of $\yen(\underline{0}), \ldots, \yen(\underline{n})$ is by definition the smallest additive subcategory of $\mathfrak{Fun}^\circ(\underline{TL}, \underline{Vect})$ containing all direct summands of $\yen(\underline{0}), \ldots, \yen(\underline{n})$ and that is closed under composition by isomorphisms in $\mathfrak{Fun}^\circ(\underline{TL}, \underline{Vect})$.

Proof. The proof proceeds by induction. For $n = 1$, define $X_0 = \mathrm{Im}(\mathrm{id}_{\yen(\underline{0})}) = \yen(\underline{0})$ and $X_1 = \mathrm{Im}(\mathrm{id}_{\yen(\underline{1})}) = \yen(\underline{1})$. Then

$$X_0 \otimes X_1 = \mathrm{Im}(\mathrm{id}_{\yen(\underline{0})} \otimes \mathrm{id}_{\yen(\underline{1})}) = \mathrm{Im}(\mathrm{id}_{\yen(\underline{1})}) = X_1 \ ,$$

as required. Furthermore, $\mathrm{End}(X_0) \cong \mathrm{End}(\underline{0}) \cong \mathbb{C}$ and $\mathrm{End}(X_1) \cong \mathrm{End}(\underline{1}) \cong \mathbb{C}$, whence X_0 and X_1 are bricks. By parity, $\mathrm{Hom}(X_0, X_1) = 0 = \mathrm{Hom}(X_1, X_0)$. Hence, X_0 and X_1 form a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ generated by (direct summands of) $\yen(\underline{0})$ and $\yen(\underline{1})$.

Let $n = 2$ and consider $\mathrm{End}(X_1 \otimes X_1) = \mathrm{End}(\yen(\underline{2})) \cong \mathrm{End}(\underline{2})$. Recall that $\mathrm{End}(\underline{2})$ has a filtration

$$\mathcal{F}_0(\underline{2}) \subset \mathcal{F}_1(\underline{2}) \subset \mathcal{F}_2(\underline{2}) = \mathrm{End}(\underline{2}) .$$

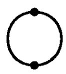
In particular, composition of morphisms defines a surjection

$$\mathrm{Hom}(\underline{2}, \underline{p}) \otimes_{\mathbb{C}} \mathrm{Hom}(\underline{p}, \underline{2}) \longrightarrow \mathcal{F}_p(\underline{2}) .$$

Consider therefore $\mathrm{Hom}(\underline{2}, \underline{0})$. This has a $\mathbb{C}$-basis given by the pairing

$\phi =$  .

Likewise $\mathrm{Hom}(\underline{0}, \underline{2})$ has a $\mathbb{C}$-basis given by ϕ°. In particular, $\mathcal{F}_0(\underline{2})$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$. Moreover,

$\phi^\circ\phi =$  $= [2]_q$,

whence, provided $[2]_q \neq 0$, the morphism $\frac{1}{[2]_q}\phi\phi^\circ$ is an idempotent in $\mathrm{End}(\underline{2})$. By parity $\mathcal{F}_1(\underline{2}) = \mathcal{F}_0(\underline{2})$. Define e_2 to be the complementary idempotent in $\mathrm{End}(\underline{2})$:

$$e_2 = \mathrm{id}_{\underline{2}} - \frac{1}{[2]_q}\phi\phi^\circ .$$

Then e_2 satisfies

$$e_2\mathcal{F}_1(\underline{2}) = 0 = \mathcal{F}_1(\underline{2})e_2 \tag{5.13}$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_2(\underline{2})}{\mathcal{F}_1(\underline{2})} \cong \mathbb{C}$. Define $X_2 = \mathrm{Im}((e_2)_*)$. Then

$$\mathrm{End}(X_2) \cong e_2 \, \mathrm{End}(\underline{2}) e_2 = \mathbb{C}e_2 ,$$

whence X_2 is a brick. By construction

$$X_1 \otimes X_1 \cong X_0 \oplus X_2 ,$$

with

$$\begin{aligned}\mathrm{Hom}(X_0, X_1 \otimes X_1) &= \mathbb{C}(\phi^\circ)_* \,,\\ \mathrm{Hom}(X_2, X_1 \otimes X_1) &= \mathbb{C}(e_2)_* \,,\end{aligned}$$

by Proposition 5.3.6. Furthermore, (5.13) ensures that

$$\mathrm{Hom}(X_2, X_p) = 0 = \mathrm{Hom}(X_p, X_2) \,, \qquad p = 0, 1 \,.$$

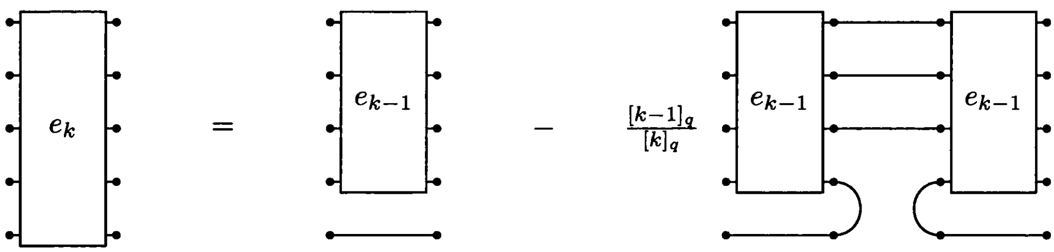
Explicitly, $\mathrm{Hom}(X_2, X_1) = 0$ by parity so it remains to consider $\phi \in \mathrm{Hom}(X_2, X_0)$. Proposition 5.3.6 ensures that $\phi = f_*$ for some $f \in \mathrm{Hom}(\underline{2}, \underline{0})$ satisfying $e_2 f = f$. The creation operator $\subset \colon \underline{0} \to \underline{2}$ defines an injection,

$$(\subset)_* \colon \mathrm{Hom}(\underline{2}, \underline{0}) \longrightarrow \mathcal{F}_0(\underline{2}) \,.$$

But $(\subset)_*(f) = f\subset \in \mathcal{F}_0(\underline{2})$ satisfies $e_2 f\subset = f\subset$, whence $f\subset = 0$ by (5.13). Likewise, $\mathrm{Hom}(X_p, X_2) = 0$ for $p = 0, 1$.

Finally, ¥$(\underline{2}) \cong X_0 \oplus X_2$, whence X_0, X_1, X_2 is a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ generated by direct summands of ¥$(\underline{0})$, ¥$(\underline{1})$, ¥$(\underline{2})$.

As an induction hypothesis, fix n and suppose that $[k]_q \neq 0$ for each $1 \leq k \leq n$. Suppose further that the maps defined inductively by



are idempotents. Finally, suppose that the objects in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ defined by $X_k = \mathrm{Im}((e_k)_*)$ satisfy the hypotheses of the theorem.

It is an easy induction to show that the idempotents e_k satisfy $e_k = (e_k)^\circ$ and that the identity map $\mathrm{id}_{\underline{k}}$ appears with coefficient 1 in the expression of e_k as a linear combination of planar Brauer diagrams. It is also clear that

$$e_k(e_{k-1} \otimes 1) = e_k = (e_{k-1} \otimes 1)e_k \,. \qquad (5.14)$$

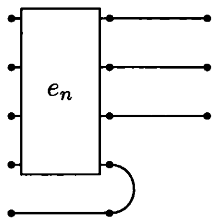
Consider $\mathrm{End}(X_n \otimes X_1) \cong (e_n \otimes 1)\,\mathrm{End}(\underline{n+1})(e_n \otimes 1)$. Recall that $\mathrm{End}(\underline{n+1})$ has a filtration by the number of through-strings. This induces the following filtration of $(e_n \otimes 1)\,\mathrm{End}(\underline{n+1})(e_n \otimes 1)$:

$$0 = (e_n \otimes 1)\mathcal{F}_0(\underline{n+1})(e_n \otimes 1) \subset \cdots \subset (e_n \otimes 1)\mathcal{F}_{n+1}(\underline{n+1})(e_n \otimes 1) \cong \mathrm{End}(X_n \otimes X_1)\ .$$

In particular, composition of morphisms defines a surjection

$$(e_n \otimes 1)\,\mathrm{Hom}(\underline{n+1}, \underline{p}) \otimes_{\mathbb{C}} \mathrm{Hom}(\underline{p}, \underline{n+1})(e_n \otimes 1) \longrightarrow (e_n \otimes 1)\mathcal{F}_p(\underline{n+1})(e_n \otimes 1)\ .$$

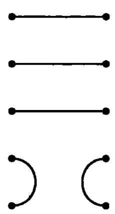
Consider the case $p = n-1$. To see that $(e_n \otimes 1)\,\mathrm{Hom}(\underline{n+1}, \underline{n-1})$ is one-dimensional, notice that every diagram from $\underline{n+1}$ to $\underline{n-1}$ must contain at least one pairing of adjacent dots on the left edge. However, the only non-zero composition of $e_n \otimes 1$ with a diagram from $\underline{n+1}$ to $\underline{n-1}$ is the following:

$\phi =$ 

.

This is because all other diagrams from $\underline{n+1}$ to $\underline{n-1}$ admit a factorisation through $\underline{n-2} \otimes \underline{1}$. However, by hypothesis, $\underline{n-2}$ has a direct sum decomposition into summands of the form X_p with $p \leq n-2$ and $\mathrm{Hom}(X_n, X_p) = 0$ whenever $p < n$, whence $e_n\,\mathrm{Hom}(\underline{n}, \underline{n-2}) = 0$. Strictly speaking, ϕ could also be the zero map, however it will emerge shortly that this is not possible. For the time being, assume that $\phi \neq 0$.

Thus $(e_n \otimes 1)\,\mathrm{Hom}(\underline{n+1}, \underline{n-1})$ has a $\mathbb{C}$-basis given by ϕ. Likewise, $\mathrm{Hom}(\underline{n-1}, \underline{n+1})\,(e_n \otimes 1)$ has a $\mathbb{C}$-basis given by ϕ° and therefore $(e_n \otimes 1)\mathcal{F}_p(\underline{n+1})(e_n \otimes 1)$ is spanned by $\phi\phi^\circ$. It is shown that $\phi\phi^\circ$ is not the zero map. To achieve this, it is established that the diagram

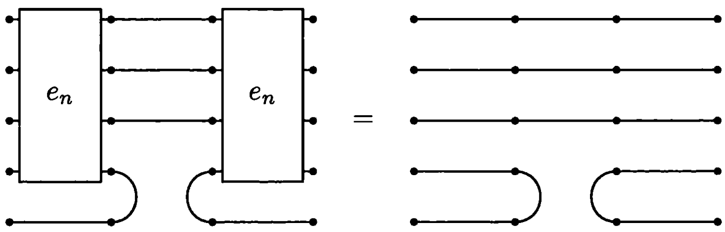
$\psi =$ 

occurs with coefficient 1 in the expression of $\phi\phi^\circ$ as a linear combination of planar

Brauer diagrams. Notice that this also settles the issue of $\phi \neq 0$.

Recall that the identity map occurs with coefficient 1 in the expression of e_n as a linear combination of diagrams. Now, considering only the component of the product $\phi\phi^\circ$ obtained by taking the identity component in both expressions of e_n yields

$\phi\phi^\circ =$

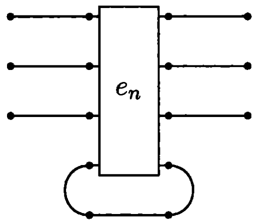


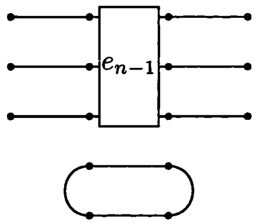
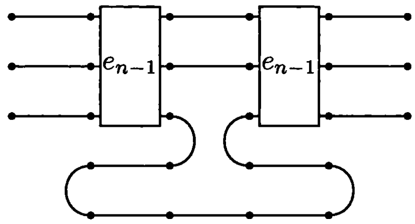
$+$ other terms

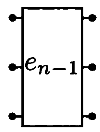
$$= \quad \psi + \text{other terms.}$$

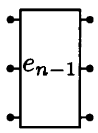
It remains to show that ψ cannot be obtained any other way in the product $\phi\phi^\circ$. Let γ be a diagram from $\underline{n}$ to $\underline{n}$ and suppose further that γ is not the identity map. Then γ induces a pairing of dots on the left edge of the rectangle. This pairing is invariant under composition of maps on the right of γ. In particular, terms in the product $\phi\phi^\circ$ resulting from taking the γ component of the left-hand e_n cannot yield multiples of the diagram ψ. Similarly, terms in the product $\phi\phi^\circ$ resulting from taking the γ° component of the right-hand e_n cannot yield multiples of ψ.

It has been established therefore, that $(e_n \otimes 1)\mathcal{F}_{n-1}(\underline{n+1})(e_n \otimes 1)$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$. Consider the following calculation:

$\phi^\circ \phi \quad =$ 

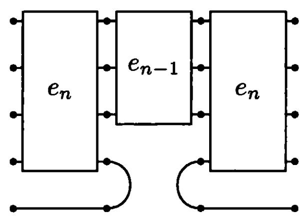
$=$  $- \ \frac{[n-1]_q}{[n]_q}$ 

$= \quad \left([2]_q - \frac{[n-1]_q}{[n]_q}\right)$ 

$= \quad \frac{[n+1]_q}{[n]_q}$  .

Then, provided $[n+1]_q \neq 0$ and using (5.14),

$$\left(\tfrac{[n]_q}{[n+1]_q}\phi\phi^\circ\right)\left(\tfrac{[n]_q}{[n+1]_q}\phi\phi^\circ\right) \quad = \quad \tfrac{[n]_q}{[n+1]_q}\phi e_{n-1}\phi^\circ$$

$= \quad \frac{[n]_q}{[n+1]_q}$ 

$$= \quad \tfrac{[n]_q}{[n+1]_q}\phi\phi^\circ \quad .$$

Hence $\frac{[n]_q}{[n+1]_q}\phi\phi^\circ$ is an idempotent in $(e_n \otimes 1)\,\mathrm{End}(\underline{n+1})(e_n \otimes 1)$. Define e_{n+1} to be the complementary idempotent in $(e_n \otimes 1)\operatorname{End}(\underline{n+1})(e_n \otimes 1)$:

$$e_{n+1} = e_n \otimes 1 - \frac{[n]_q}{[n+1]_q}\phi\phi^\circ .$$

Then e_{n+1} satisfies

$$e_{n+1}\mathcal{F}_p(\underline{n+1}) = 0 = \mathcal{F}_p(\underline{n+1})e_{n+1} , \qquad \forall p < n+1 \tag{5.15}$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_{n+1}(\underline{n+1})}{\mathcal{F}_n(\underline{n+1})} \cong \mathbb{C}$ because the identity map occurs with coefficient 1 in the expression of e_{n+1} as a linear combination of planar Brauer diagrams. Define $X_{n+1} = \operatorname{Im}((e_{n+1})_*)$. Then

$$\operatorname{End}(X_{n+1}) \cong e_{n+1}\operatorname{End}(\underline{n+1})e_{n+1} = \mathbb{C}e_{n+1} ,$$

whence X_{n+1} is a brick. By construction,

$$X_n \otimes X_1 \cong X_{n+1} \oplus X_{n-1} ,$$

with

$$\begin{aligned} \operatorname{Hom}(X_{n-1}, X_n \otimes X_1) &= \mathbb{C}(\phi^\circ)_* , \\ \operatorname{Hom}(X_{n+1}, X_n \otimes X_1) &= \mathbb{C}(e_{n+1})_* . \end{aligned}$$

Furthermore, (5.15) ensures that

$$\operatorname{Hom}(X_{n+1}, X_p) = 0 = \operatorname{Hom}(X_p, X_{n+1}) , \qquad \forall p < n+1 .$$

One way of seeing this explicitly is as follows: suppose $\alpha \in \operatorname{Hom}(X_{n+1}, X_p)$ for some $p < n+1$. In particular, $\alpha = f_*$ for some $f \in \operatorname{Hom}(\underline{n+1}, \underline{p})$ satisfying $e_{n+1}fe_p = f$. If p and $n+1$ do not have the same parity then $f = 0$. Suppose therefore that p and $n+1$ do have the same parity. Let χ denote the map

$$\mathrm{id}_{\underline{p}} \otimes \overbrace{\cup \otimes \cdots \otimes \cup}^{\frac{n+1-p}{2}\text{ times}} : \underline{p} \longrightarrow \underline{n+1} .$$

Then $\chi_* \colon \operatorname{Hom}(\underline{n+1}, \underline{p}) \to \mathcal{F}_p(\underline{n+1})$ and is injective (provided $[2]_q \neq 0$) since

$$\chi\chi^\circ = [2]_q^{\frac{n+1-p}{2}} \mathrm{id}_{\underline{p}} .$$

But $\chi_*(f) = f\chi \in \mathcal{F}_p(\underline{n+1})$ satisfies $e_{n+1}f\chi = f\chi$. Hence $f\chi = 0$ by (5.15) and therefore $\alpha = 0$. Likewise, $\mathrm{Hom}(X_p, X_{n+1}) = 0$ for every $p < n+1$.

Finally, $¥(\underline{n+1}) = ¥(\underline{n}) \otimes ¥(\underline{1})$ and a simple induction using the fusion rule proves that $X_0, \ldots, X_{n+1}$ is a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ generated by direct summands of $¥(\underline{0}), \ldots, ¥(\underline{n+1})$. This completes the proof of the theorem. □

It will be shown later that for singular values of q the bricks X_p have interesting subobjects. However, for generic q the X_p are simple.

Lemma 5.4.4. *Suppose $[k]_q \neq 0$ for all $1 \leq k \leq n$ and let η be a non-zero natural transformation $X_p \to ¥(\underline{n})$. Then there exists a natural transformation $\chi\colon ¥(\underline{n}) \to X_p$ satisfying $\chi \circ \eta = \mathrm{id}_{X_p}$. Similarly, there is a right inverse to every non-zero morphism $¥(\underline{n}) \to X_p$.*

Proof. Let η be a non-zero transformation $X_p \to ¥(\underline{n})$. Choose a direct sum decomposition for $¥(\underline{n})$ of the form

$$¥(\underline{n}) = \bigoplus_k X_{m_k} ,$$

including the inclusion maps ι_k and the projection maps π_k. Then X_p is a direct summand in $¥(\underline{n})$ since

$$\eta = \eta\,\mathrm{id}_{¥(\underline{n})} = \sum_k \eta\pi_k\iota_k$$

and $\mathrm{Hom}(X_p, X_{m_k}) = 0$ whenever $p \neq m_k$. In particular, there exists some index k for which $\eta\pi_k \neq 0$. However, $\eta\pi_k \in \mathrm{End}(X_p) \cong \mathbb{C}$, hence there is a non-zero $\lambda \in \mathbb{C}$ such that $\eta\pi_k = \lambda\,\mathrm{id}_{X_p}$. Define $\chi = \frac{1}{\lambda}\pi_k$.

The proof of the second part of the lemma is analogous. □

Theorem 5.4.5. *For generic q, the functors X_p are simple in $\mathfrak{Fun}^\circ(\underline{TL}, \underline{Vect})$.*

Proof. Suppose $\mathcal{G}$ is a subobject of X_p. Then there is an exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{\mu} X_p \xrightarrow{\epsilon} \mathcal{H} \longrightarrow 0 .$$

In particular, for each $\underline{n}$, evaluating the above exact sequence of functors yields a short exact sequence of vector spaces:

$$0 \longrightarrow \mathcal{G}(\underline{n}) \xrightarrow{\mu_{\underline{n}}} X_p(\underline{n}) \xrightarrow{\epsilon_{\underline{n}}} \mathcal{H}(\underline{n}) \longrightarrow 0 .$$

Suppose that $\mathcal{H}(\underline{p}) = 0$. For $\phi \in X_p(\underline{n})$ the following diagram commutes:

$$\begin{tikzcd}
X_p(\underline{p}) \arrow[r, "\epsilon_{\underline{p}}"] \arrow[d, "\phi^*"'] & \mathcal{H}(\underline{p}) \arrow[d, "\mathcal{H}(\phi)"] \\
X_p(\underline{n}) \arrow[r, "\epsilon_{\underline{n}}"'] & \mathcal{H}(\underline{n})
\end{tikzcd} .$$

In particular,

$$\epsilon_{\underline{n}}(\phi) = \epsilon_{\underline{n}} \circ \phi^*(e_p) = \mathcal{H}(\phi) \circ \epsilon_{\underline{p}}(e_p) = 0 \, .$$

But cokernels in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ are defined pointwise, whence $\mathcal{H}(\underline{n}) = 0$ for every $\underline{n}$. Consequently, $\mathcal{G} \cong X_p$.

It remains to consider the case $\mathcal{H}(\underline{p}) \neq 0$. The following sequence is exact:

$$0 \longrightarrow \mathcal{G}(\underline{p}) \longrightarrow X_p(\underline{p}) \longrightarrow \mathcal{H}(\underline{p}) \longrightarrow 0 \, .$$

In particular,

$$X_p(\underline{p}) = \mathrm{Hom}(X_p, \yen(\underline{p})) \cong \{\phi \colon \underline{p} \to \underline{p} \; : \; \phi e_p = \phi\} \cong \mathbb{C} \, .$$

Now, $\mathcal{H}(\underline{p}) \neq 0$ implies that $\mathcal{G}(\underline{p}) = 0$. Furthermore, $X_p(\underline{n}) = 0$ for every $n < p$, whence $\mathcal{G}(\underline{n}) = 0$. Fix $n > p$ and suppose that $\mathcal{G}(\underline{n}) \neq 0$. Then n and p have the same parity (else $X_p(\underline{n}) = 0$). Moreover, there must exist $v \in \mathcal{G}(\underline{n})$ such that $\mu_{\underline{n}}(v) \neq 0$, since $\mathcal{G}$ is a kernel. Such a $\mu_{\underline{n}}(v)$ is a morphism $\underline{n} \to \underline{p}$ satisfying $\mu_{\underline{n}}(v) e_p = \mu_{\underline{n}}(v)$.

Since q is generic, Lemma 5.4.4 ensures that there is a natural transformation $\iota_* \colon X_p \to \yen(\underline{n})$ such that $0 \neq \iota\mu_{\underline{n}}(v)$. The following diagram is commutative:

$$\begin{tikzcd}
\mathcal{G}(\underline{n}) \arrow[r, "\mu_{\underline{n}}"] \arrow[d, "\mathcal{G}(\iota)"'] & X_p(\underline{n}) \arrow[d, "\iota^*"] \\
\mathcal{G}(\underline{p}) \arrow[r, "\mu_{\underline{p}}"'] & X_p(\underline{p})
\end{tikzcd} .$$

For $v \in \mathcal{G}(\underline{n})$ fixed earlier, this yields

$$0 \neq \iota\mu_{\underline{n}}(v) = \iota^* \circ \mu_{\underline{n}}(v) = \mu_{\underline{p}} \circ \mathcal{G}(\iota)(v) = 0 \ ,$$

since $\mathcal{G}(\underline{p}) = 0$.

In conclusion, if $\mathcal{H}(\underline{p}) = 0$ then $\mathcal{G} \cong X_p$ and if $\mathcal{H}(\underline{p}) \neq 0$ then $\mathcal{G} = 0$. Hence X_p is a simple object in $\mathfrak{Fun}^{\circ}(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$. □

Corollary 5.4.6. *For generic q, the abelian subcategory of $\mathfrak{Fun}^{\circ}(\underline{TL}, \underline{Vect})$ generated by $\yen(\underline{TL})$ is semisimple.*

The abelian subcategory generated by $\yen(\underline{\mathrm{TL}})$ is defined to be the smallest abelian subcategory of $\mathfrak{Fun}^{\circ}(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ containing $\yen(\underline{\mathrm{TL}})$ and that is closed under composition by isomorphisms in $\mathfrak{Fun}^{\circ}(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$.

Let $\mathfrak{F} \in \mathfrak{Fun}^{\circ}(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$. Then $\mathrm{Hom}(\mathfrak{F}, \yen(\underline{n}))$ is a $\mathbb{C}$-vector space for each n. In fact, $\mathrm{Hom}(\mathfrak{F}, \yen(\underline{n}))$ is a TL_n-module. Let $\phi \in \mathrm{End}(\underline{n})$. Then $\phi_* \in \mathrm{End}(\yen(\underline{n}))$ and hence for $\eta \in \mathrm{Hom}(\mathfrak{F}, \yen(\underline{n}))$, the composition $\phi_* \circ \eta$ is also a natural transformation $\mathfrak{F} \to \yen(\underline{n})$.

Henceforth, to coincide with composition in the Temperley-Lieb category, natural transformations will also compose left to right.

Proposition 5.4.7. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq n$. Then the TL_n-module* $\mathrm{Hom}(X_p, \yen(\underline{n}))$ *is simple.*

Proof. Let η and χ be non-zero natural transformations $X_p \to \yen(\underline{n})$. By Lemma 5.4.4 there exists a natural transformation $\phi \in \mathrm{Hom}(\yen(\underline{n}), X_p)$ satisfying $\eta\phi = \mathrm{id}_{X_p}$. Define $\psi = \phi\chi \in \mathrm{End}(\yen(\underline{n}))$. Then $\psi = f_*$ for some morphism $f \in \mathrm{End}(\underline{n})$ and

$$\eta \cdot f = \eta\psi = \chi \ .$$

Consequently, every non-zero natural transformation $\eta \in \mathrm{Hom}(X_p, \yen(\underline{n}))$ generates $\mathrm{Hom}(X_p, \yen(\underline{n}))$ as a TL_n-module. Hence, $\mathrm{Hom}(X_p, \yen(\underline{n}))$ is simple. □

Define the functor $X_p \otimes V$ as

$$X_p \otimes V = (\yen(\underline{p}) \otimes V)^{e_p \otimes 1} \ .$$

Then it is easily shown, with the aid of Propositions 5.3.5 and 5.3.7, that

$$\begin{aligned} \operatorname{Hom}(¥(\underline{n}), X_p \otimes V) &\cong (e_p \otimes 1)_* \operatorname{Hom}(¥(\underline{n}), ¥(\underline{p}) \otimes V) \cong (e_p)_* \operatorname{Hom}(\underline{n}, \underline{p}) \otimes V , \\ \operatorname{Hom}(X_p \otimes V, ¥(\underline{n})) &\cong (e_p \otimes 1)^* \operatorname{Hom}(¥(\underline{p}) \otimes V, ¥(\underline{n})) \cong \operatorname{Hom}(V, (e_p)^* \operatorname{Hom}(\underline{p}, \underline{n})) . \end{aligned}$$

It is now possible to present a canonical semisimple decomposition for $¥(\underline{n})$ as a Temperley-Lieb module, provided n is sufficiently small.

Theorem 5.4.8. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq n$. Then, as a $TL_n([2]_q)$-module, the object $¥(\underline{n})$ is semisimple and has a direct sum decomposition*

$$¥(\underline{n}) \cong \bigoplus_{p=0}^{n} X_p \otimes_{\mathbb{C}} \operatorname{Hom}(X_p, ¥(\underline{n})) .$$

Proof. Consider the natural map

$$\chi \colon \bigoplus_{p=0}^{n} X_p \otimes_{\mathbb{C}} \operatorname{Hom}(X_p, ¥(\underline{n})) \longrightarrow ¥(\underline{n})$$

defined as the direct sum of the evaluation maps. This is clearly a morphism of TL_n-modules. It remains to show that it is an isomorphism. Choose a direct sum decomposition for $¥(\underline{n})$ of the form

$$¥(\underline{n}) \cong \bigoplus_{k} X_{p_k} ,$$

including the inclusion maps $(\iota_k)_*$ and the projection maps $(\pi_k)_*$. For $\underline{m} \in \underline{\mathrm{TL}}$, define the map

$$\begin{aligned} \theta_{\underline{m}} \colon (¥(\underline{n}))(\underline{m}) &\longrightarrow \bigoplus_{p=0}^{n} X_p(\underline{m}) \otimes \operatorname{Hom}(X_p, ¥(\underline{n})) \\ \left(\underline{m} \xrightarrow{f} \underline{n}\right) &\longmapsto \sum_k f\pi_k \otimes (\iota_k)_* . \end{aligned}$$

Then $\chi \circ \theta = \mathrm{id}_{¥(\underline{n})}$. In particular, $\theta_{\underline{m}}$ is injective for every $\underline{m}$. It remains to show that $\theta_{\underline{m}}$ is surjective. Define $I(p)$ to be the subset of indices k such that $(\iota_k)_*$ has codomain X_p. Then $\{(\iota_k)_*\}_{k \in I(p)}$ is a spanning set for $\operatorname{Hom}(X_p, ¥(\underline{n}))$. Explicitly, for $\eta \in \operatorname{Hom}(X_p, ¥(\underline{n}))$,

$$\eta = \eta \, \mathrm{id}_{¥(\underline{n})} = \sum_k \eta(\pi_k)_*(\iota_k)_* .$$

However, $\eta(\pi_k)_* = 0$ whenever $k \notin I(p)$ and $\eta(\pi_k)_* = \lambda_k \, \mathrm{id}_{X_p}$ whenever $k \in I(p)$.

Hence

$$\eta = \sum_{k \in I(p)} \lambda_k (\iota_k)_* \, .$$

Let $f \in X_p(\underline{m})$. Then it is sufficient to show that $f \otimes (\iota_k)_* \in \mathrm{Im}(\theta_{\underline{m}})$ for each $k \in I(p)$. Consider $(\iota_k)_*(f) = f\iota_k \in (\yen(\underline{n}))(\underline{m})$. Then

$$\theta_{\underline{m}}(f\iota_k) = \sum_r f\iota_k\pi_r \otimes (\iota_r)_* = f \otimes (\iota_k)_* \, ,$$

as required. □

Before proceeding to the study of graph representations, the analysis of the Temperley-Lieb category is concluded with the introduction of a formal algebra-coalgebra pair.

5.4.3 A formal Koszul pair for generic q

For (Q, ω) a non-Dynkin symplectic quiver, it is known that the preprojective algebra is Koszul (see [BBK02]). It will be shown that this structure arises from a formal algebra-coalgebra pair existing in the functor category, which is analysed in this section (cf.[MOV06]).

Write $\delta_0 = [2]_q \in \mathbb{C}$ for the parameter δ_0 in the Temperley-Lieb category. If q is generic then Theorem 5.4.3 constructs a sequence of bricks $X_0, X_1, X_2, \ldots$. If q is singular then there is a smallest natural number $h > 0$ such that $[h]_q = 0$ and Theorem 5.4.3 builds precisely h bricks $X_0, \ldots, X_{h-1}$. In this case, define X_k to be the zero functor for all $k \geq h$. The X_k will be the graded pieces of a formal algebra in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$.

Let $X = \bigoplus_{p=0}^{\infty} X_p$ and define the multiplication maps μ by

$$\mu_{ij} = (e_{i+j})_* \colon X_i \otimes X_j \longrightarrow X_{i+j} \, ,$$

if q is generic or if $i + j < h$ in the singular case. If q is singular and if $i + j \geq h$, then define μ_{ij} to be the zero map.

It is perhaps not evident that μ_{ij} is well-defined. It is sufficient to show that $(e_i \otimes e_j)e_{i+j} = e_{i+j}$. Write $e_i \otimes e_j$ as a linear combination of planar Brauer diagrams. Then the only component not factoring through an object of degree less than $i + j$ is the component of the identity map. This has coefficient 1, whence the result by (5.15).

Likewise the maps μ form an associative product:

$$(e_{i+j} \otimes e_k)e_{i+j+k} = e_{i+j+k} = (e_i \otimes e_{j+k})e_{i+j+k} \ .$$

There is also a formal (coassociative) coalgebra given by the graded object $A = X_0 \oplus X_1 \oplus X_0$ and where the only non-trivial comultiplication is given by the creation operator Ȼ: $A_2 \to A_1 \otimes A_1$.

Theorem 5.4.9. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq n$. Then the following sequences are exact:*

$$0 \longrightarrow X_{k-2} \otimes A_2 \longrightarrow X_{k-1} \otimes A_1 \longrightarrow X_k \longrightarrow 0 \ .$$

Proof. In Theorem 5.4.3 the map e_k was defined precisely as the complementary idempotent to an explicit splitting of the inclusion $X_{k-2} \otimes A_2 \longrightarrow X_{k-1} \otimes A_1$. There is nothing left to prove. ☐

Corollary 5.4.10. *For generic q, the pair (X, A) is Koszul.*

5.5 Graph Temperley-Lieb representations

The analysis of the Temperley-Lieb categories in the previous section is exploited to understand the ways in which $\underline{\mathrm{TL}}^-$ can act on the path algebra of a quiver.

5.5.1 Definition and examples

Definition 5.5.1. *A* GRAPH TEMPERLEY-LIEB REPRESENTATION *supported by the quiver Q is a $\mathbb{C}$-linear (strict) monoidal functor $\mathfrak{F}\colon \underline{TL}^-(\delta_0) \to (\mathbb{C}Q)_0\text{-}\underline{mod}\text{-}(\mathbb{C}Q)_0$ satisfying*

$$\mathfrak{F}(\underline{1}) = (\mathbb{C}Q)_1 \ .$$

Monoidal means that $\mathfrak{F}$ respects the tensor product and tensor unit:

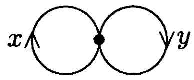
$$\begin{aligned} \mathfrak{F}(\underline{0}) &= (\mathbb{C}Q)_0 \ , \\ \mathfrak{F}(X \otimes Y) &= \mathfrak{F}(X) \otimes_{(\mathbb{C}Q)_0} \mathfrak{F}(Y) \ . \end{aligned}$$

In particular,

$$\mathfrak{F}(\underline{n}) = \mathfrak{F}(\underline{1}^{\otimes n}) = \mathfrak{F}(\underline{1})^{\otimes n} = (\mathbb{C}Q)_1^{\otimes n} = (\mathbb{C}Q)_n \ .$$

Notice that to define a graph Temperley-Lieb representation it is sufficient to specify a quiver Q and the action of the creation and annihilation operators, subject to the relations in Figure 5-2.

The fundamental example of a graph Temperley-Lieb representation is given by the quiver



Let $V = \mathbb{C}\langle x, y\rangle_1$ be the two-dimensional complex vector space spanned by x and y. Define the annihilation map by

$$\begin{array}{rcl} \mathbb{C}\langle x, y\rangle_2 & \longrightarrow & \mathbb{C} \\ x^2 & \longmapsto & 0 \\ xy & \longmapsto & 1 \\ yx & \longmapsto & -1 \\ y^2 & \longmapsto & 0 \end{array}$$

and define the creation map by

$$\begin{array}{rcl} \mathbb{C} & \longrightarrow & \mathbb{C}\langle x, y\rangle_2 \\ 1 & \longmapsto & xy - yx\,. \end{array}$$

Then these maps coincide with the natural maps between $V \otimes_{\mathbb{C}} V$ and $\Lambda^2(V) \cong \mathbb{C}$ and define a graph Temperley-Lieb representation with $[2]_q = 2 = \dim V$.

The above example belongs to a wider class of graph Temperley-Lieb representations. Let (Q, ω) be a symplectic quiver. Then the annihilation operator ϕ and the creation operator ψ defined in Section 5.1.2 extend to a graph Temperley-Lieb representation supported by Q with $[2]_q = \lambda$, where λ is the Perron-Frobenius eigenvalue of the quiver Q (but see Remark 5.1.3). The previous example is a particular case of this by setting $\omega(x, y) = 1$.

Notice that it would be possible to use a symmetric quiver to obtain a graph representation of $\underline{\mathrm{TL}}^+$, using an analogous formula to that for symplectic quivers (see Remark 5.1.2). These examples, however, provide an explanation for preferring $\underline{\mathrm{TL}}^-$ to $\underline{\mathrm{TL}}^+$. The natural invariant in the first example is $\dim V = 2$ and the creation and annihilation operators are the natural maps between $V \otimes_{\mathbb{C}} V$ and $\Lambda^2(V) \cong \mathbb{C}$. These naturally generate a representation of $\underline{\mathrm{TL}}^-$ and *not* a representation of $\underline{\mathrm{TL}}^+$. The natural extension of this example is now to symplectic quivers with Perron-Frobenius eigenvalue δ_0.

5.5.2 Extending graph representations

A graph Temperley-Lieb representation $\mathfrak{F}$ supported by Q defines naturally a monoidal functor

$$\widetilde{\mathfrak{F}}\colon \text{¥}(\underline{\mathrm{TL}}) \longrightarrow (\mathbb{C}Q)_0\text{-}\underline{\mathrm{mod}}\text{-}(\mathbb{C}Q)_0\,,$$

which can be extended to the other functors arising in the analysis of the Temperley-Lieb category. Explicitly, define $\widetilde{\mathfrak{F}}$ on $\mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, \underline{B})$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\underline{A} \otimes V, \underline{B}) \arrow[r, "\widetilde{\mathfrak{F}}"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathfrak{F}(A) \otimes V, \mathfrak{F}(B)) \arrow[d, "\cong"] \\
\mathrm{Hom}(V, \mathrm{Hom}(A, B)) \arrow[r, "\mathfrak{F}_*"'] & \mathrm{Hom}(V, \mathrm{Hom}(\mathfrak{F}(A), \mathfrak{F}(B)))
\end{tikzcd} \cdot$$

Similarly, define $\widetilde{\mathfrak{F}}$ on $\mathrm{Hom}(\underline{B}, \underline{\mathrm{A}} \otimes \mathrm{V})$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\underline{B}, \underline{A} \otimes V) \arrow[r, "\widetilde{\mathfrak{F}}"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathfrak{F}(B), \mathfrak{F}(A) \otimes V) \arrow[d, "\cong"] \\
\mathrm{Hom}(B, A) \otimes V \arrow[r, "\mathfrak{F} \otimes 1"'] & \mathrm{Hom}(\mathfrak{F}(B), \mathfrak{F}(A)) \otimes V
\end{tikzcd} \cdot$$

Suppose $\widetilde{\mathfrak{F}}$ has been defined on W and Z. If $e \in \mathrm{End}(Z)$ is an idempotent then $\widetilde{\mathfrak{F}}$ can be extended to $\mathrm{Hom}(W, Z^e)$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(W, Z^e) \arrow[r, "\widetilde{\mathfrak{F}}"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathfrak{F}(W), \mathrm{Im}(\mathfrak{F}(e))) \arrow[d, "\cong"] \\
\mathrm{Hom}(W, Z)e \arrow[r, "\mathfrak{F}"'] & \mathrm{Hom}(\mathfrak{F}(W), \mathfrak{F}(Z))\mathfrak{F}(e)
\end{tikzcd}$$

and to $\mathrm{Hom}(Z^e, W)$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(Z^e, W) \arrow[r, "\widetilde{\mathfrak{F}}"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathrm{Im}(\mathfrak{F}(e)), \mathfrak{F}(W)) \arrow[d, "\cong"] \\
e\,\mathrm{Hom}(Z, W) \arrow[r, "\mathfrak{F}"'] & \mathfrak{F}(e)\,\mathrm{Hom}(\mathfrak{F}(Z), \mathfrak{F}(W))
\end{tikzcd} .$$

It is simple, if a little tedious, to check that $\mathfrak{F}$ is indeed functorial and monoidal. Henceforth, all graph Temperley-Lieb representations $\mathfrak{F}$ will be understood to have been augmented in this fashion. They will continue to be denoted by $\mathfrak{F}$.

A graph Temperley-Lieb representation $\mathfrak{F}$ defines, for each $n \geq 2$, a representation of the algebra $TL_n([2]_q)$ supported by $(\mathbb{C}Q)_n$. For generic values of q, (that is, $[k]_q \neq 0$ for every $k > 0$,) Theorem 5.4.8 ensures that these representations are semisimple and have the decomposition formula

$$(\mathbb{C}Q)_n \cong \bigoplus_{p=0}^{n} \mathfrak{F}(X_p) \otimes_{\mathbb{C}} \mathrm{Hom}(X_p, \yen(\underline{n})) ,$$

where for each p, the object $\mathfrak{F}(X_p)$ is the multiplicity in $(\mathbb{C}Q)_n$ of the simple $TL_n([2]_q)$-module $\mathrm{Hom}(X_p, \yen(\underline{n}))$. The $\mathfrak{F}(X_p)$ carry the $(\mathbb{C}Q)_0$-bimodule structure on $(\mathbb{C}Q)_n$ and for each pair $i, j \in Q_0$, the direct summand $e_i\mathfrak{F}(X_p)e_j$ must have non-negative dimension. Notice that, since $\mathfrak{F}$ is a $\mathbb{C}$-linear monoidal functor, the fusion rule for the simple objects X_p is preserved, that is,

$$\mathfrak{F}(X_p) \otimes_{(\mathbb{C}Q)_0} (\mathbb{C}Q)_1 = \mathfrak{F}(X_{p-1}) \oplus \mathfrak{F}(X_{p+1}) , \qquad p \geq 1 .$$

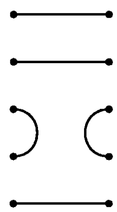
5.5.3 Koszul pairs from generic representations

Fix a quiver Q together with a graph Temperley-Lieb representation $\mathfrak{F}$ with $\delta_0 = [2]_q$. Recall that for singular q, the functor X_p is defined to be the zero functor if $p \geq h$. Define

$$\begin{aligned} \Sigma &= \bigoplus_{p=0}^{\infty} \mathfrak{F}(X_p) \ , \\ \Lambda &= \mathfrak{F}(X_0) \oplus \mathfrak{F}(X_1) \oplus \mathfrak{F}(X_0) \ , \end{aligned}$$

effectively the image under $\mathfrak{F}$ of the formal algebra and the formal coalgebra defined in Section 5.4.3.

Recall (Definition 5.1.4) that the space of essential paths of length n, $\mathrm{EssPath}_n$, consists of precisely the paths of length n that lie in the kernel of (the image under $\mathfrak{F}$ of) every Temperley-Lieb operator

$U_i \quad =$  $, \qquad 1 \leq i \leq n-1 \ .$

Proposition 5.5.2. *Suppose $[k]_q \neq 0$ for $1 \leq k \leq n$. Then Σ_n coincides with the space of essential paths* $\mathrm{EssPath}_n$.

Proof. Recall that $\Sigma_n = \mathfrak{F}(X_n) = \mathrm{Im}\,\mathfrak{F}(e_n)$ and $e_n U_i = 0$ for every $1 \leq i \leq n-1$. Hence $\Sigma_n \subset \mathrm{EssPath}_n$. For the reverse inclusion, let $p \in \mathrm{EssPath}_n$. Then $p = (p)(\mathfrak{F}(e_n))$ since the only component of e_n that does not factor through one of the U_i is the identity diagram, which occurs with coefficient 1. Thus $p \in \mathrm{Im}\,\mathfrak{F}(e_n) = \Sigma_n$ as required. □

Moreover, Σ and Λ inherit an algebra and coalgebra structure respectively and satisfy:

Corollary 5.5.3. *If q is generic then (Σ, Λ) is a Koszul pair.*

Consider the two-sided ideal $\mathfrak{I} \subset \mathbb{C}Q$ generated by $\mathrm{Im}(\mathfrak{C})$. Define the quotient algebra Π via the short exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathbb{C}Q \longrightarrow \Pi \longrightarrow 0 \ .$$

Notice that $\mathfrak{I}$ is a homogeneous ideal, hence Π inherits the grading from $\mathbb{C}Q$. It will be shown that the graded pieces of Π can be identified with the spaces $\Sigma_p = \mathfrak{F}(X_p)$

and that, for generic q, the algebras Π and Σ are isomorphic. In particular, (Π, Λ) is a Koszul pair.

Remark 5.5.4. *For generic q, the results follow immediately from the fact that Koszul algebras are quadratic ([BGS96]). It is illuminating, however, to see an explicit argument.*

There is an obvious candidate for such an isomorphism $\Sigma \dashrightarrow \Pi$, given by the composite of the inclusion map $j\colon \Sigma \to \mathbb{C}Q$ and the quotient map $\pi\colon \mathbb{C}Q \to \Pi$.

Lemma 5.5.5. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq n$ and let e_n denote the idempotent defining X_n. Then $\pi_n \circ \mathfrak{F}(e_n) = \pi_n$.*

Proof. Recall that $e_n = \mathrm{id}_{\underline{n}} + \phi$, where ϕ is a linear combination of planar Brauer diagrams factoring through an object of degree strictly less than n. In particular, any such diagram must factor through a tensor extension of $\subset$. Hence, $\mathrm{Im}(\mathfrak{F}(\phi)) \in \mathfrak{I}_n$ and now for $a \in (\mathbb{C}Q)_n$,

$$\pi_n \circ \mathfrak{F}(e_n)(a) = \pi_n(a) + \pi_n \circ \mathfrak{F}(\phi)(a) = \pi_n(a)\ .$$

$\square$

Denote by $\Pi^{(\leq n)}$, the truncation of the algebra Π after degree n.

Proposition 5.5.6. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq n$. Then the induced map $\Sigma \to \Pi^{(\leq n)}$ is an algebra homomorphism.*

Proof. Consider the diagram

$$\begin{tikzcd}
\Sigma_r \otimes \Sigma_s \arrow[r, "m"] \arrow[d, "j \otimes j"'] & \Sigma_{r+s} \arrow[d, "j"'] & \\
(\mathbb{C}Q)_r \otimes (\mathbb{C}Q)_s \arrow[r, "\mu"] \arrow[dr, "\pi \otimes \pi"'] & (\mathbb{C}Q)_{r+s} \arrow[u, dashed, bend right, "p"'] \arrow[dr, "\pi"] & \\
& \Pi_r \otimes \Pi_s \arrow[r, "\nu"] & \Pi_{r+s}
\end{tikzcd}$$

where m, μ and ν denote the products in Σ, $\mathbb{C}Q$ and Π respectively. If $r + s \geq n + 1$ then there is nothing to prove. Let $r + s \leq n$ and denote by p the retraction satisfying

$j \circ p = \mathfrak{F}(e_{r+s})$. Notice that, whilst the lower square commutes, the upper square does not. However, $m = p \circ \mu \circ (j \otimes j)$ and now, using Lemma 5.5.5,

$$\begin{aligned}
\pi \circ j \circ m &= \pi \circ j \circ p \circ \mu \circ (j \otimes j) \\
&= \pi \circ \mathfrak{F}(e_{r+s}) \circ \mu \circ (j \otimes j) \\
&= \pi \circ \mu \circ (j \otimes j) \\
&= \nu \circ (\pi \otimes \pi) \circ (j \otimes j) ,
\end{aligned}$$

as required. □

Corollary 5.5.7. *For generic q, the map $\pi \circ j \colon \Sigma \to \Pi$ is an algebra homomorphism.*

Before proceeding to the proof that this map is generically an isomorphism, consider the following. Fix $n > 2$. For $1 \leq i \leq n-1$ define

$$C_i = \overbrace{1 \otimes \cdots \otimes 1}^{i-1 \text{ times}} \otimes \mathfrak{c} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-1-i \text{ times}} .$$

Then

$$\begin{aligned}
\Pi_n &= \frac{(\mathbb{C}Q)_n}{\mathfrak{I}_n} \\
&= \frac{(\mathbb{C}Q)_n}{\mathrm{Im}(C_1) + \cdots + \mathrm{Im}(C_{n-1})} \\
&= \frac{(\mathbb{C}Q)_n}{\mathfrak{I}_{n-1} \otimes (\mathbb{C}Q)_1 + (\mathbb{C}Q)_{n-2} \otimes \mathrm{Im}(\mathfrak{c})} .
\end{aligned}$$

Taking the quotient by $\mathfrak{I}_{n-1} \otimes (\mathbb{C}Q)_1$ yields an exact sequence

$$(\mathbb{C}Q)_{n-2} \otimes \mathrm{Im}(\mathfrak{c}) \longrightarrow \Pi_{n-1} \otimes (\mathbb{C}Q)_1 \longrightarrow \Pi_n \longrightarrow 0 .$$

However, $(\mathbb{C}Q)_{n-2} \otimes \mathrm{Im}(\mathfrak{c}) \cong (\mathbb{C}Q)_{n-2} \otimes \Lambda_2$ is not the kernel of this sequence and there is a factorisation

$$
\begin{tikzcd}
0 \\
\Pi_{n-2} \otimes \Lambda_2 \arrow[u] \arrow[dr, dashed] \\
(\mathbb{C}Q)_{n-2} \otimes \Lambda_2 \arrow[u] \arrow[r] & \Pi_{n-1} \otimes \Lambda_1 \arrow[r] & \Pi_n \arrow[r] & 0 \\
\mathfrak{I}_{n-2} \otimes \Lambda_2 \arrow[u] \\
0 \arrow[u]
\end{tikzcd}
,
$$

Hence there is an exact sequence

$$
\Pi_{n-2} \otimes \Lambda_2 \longrightarrow \Pi_{n-1} \otimes \Lambda_1 \longrightarrow \Pi_n \longrightarrow 0 \, ,
$$

where the maps are precisely the Koszul differential on $\Pi_\bullet \otimes \Lambda_\bullet$.

In particular, provided $[k]_q \neq 0$ for every $2 \leq k \leq n$, the algebra homomorphism $\Sigma \to \Pi^{(\leq n)}$ induces a map of exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & \Sigma_{k-2} \otimes \Lambda_2 \arrow[r] \arrow[d] & \Sigma_{k-1} \otimes \Lambda_1 \arrow[r] \arrow[d] & \Sigma_k \arrow[r] \arrow[d] & 0 \\
 & \Pi_{k-2} \otimes \Lambda_2 \arrow[r] & \Pi_{k-1} \otimes \Lambda_1 \arrow[r] & \Pi_k \arrow[r] & 0 \, .
\end{tikzcd}
$$

Theorem 5.5.8. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq n$. Then the map $\Sigma_n \to \Pi_n$ is an isomorphism.*

Proof. Clearly $\Sigma_0 = (\mathbb{C}Q)_0 = \Pi_0$ and $\Sigma_1 = (\mathbb{C}Q)_1 = \Pi_1$ and now the result follows from the 5-lemma by induction. □

There are a number of immediate corollaries:

Corollary 5.5.9. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq n$. Then the truncated algebras $\Sigma^{(\leq n)}$ and $\Pi^{(\leq n)}$ are isomorphic.*

Corollary 5.5.10. *For generic q, the algebras Σ and Π are isomorphic.*

Corollary 5.5.11. *Suppose that $[k]_q \neq 0$ for every $2 \leq k \leq n$. Then the following sequences are exact:*

$$0 \longrightarrow \Pi_{k-2} \otimes \Lambda_2 \longrightarrow \Pi_{k-1} \otimes \Lambda_1 \longrightarrow \Pi_k \longrightarrow 0\,.$$

Corollary 5.5.12. *For generic q, the pair (Π, Λ) is Koszul.*

Finally, to conclude this section:

Corollary 5.5.13. *Let (Q, ω) be a non-Dynkin symplectic quiver and denote the preprojective algebra by Π. Then the pair (Π, Λ) is Koszul.*

Proof. Consider the graph representation defined in Section 5.1.2. Then $[2]_q$ is the Perron-Frobenius eigenvalue of Q and since Q is non-Dynkin, $[2]_q \geq 2$ hence q is generic. The preprojective algebra coincides with $\frac{\mathbb{C}Q}{\mathfrak{J}}$, where $\mathfrak{J}$ is the ideal in $\mathbb{C}Q$ generated by $\mathrm{Im}(\mathfrak{C})$. Hence the pair (Π, Λ) is Koszul. $\square$

In the remainder of this chapter, the singular case is analysed in more detail.

5.6 The reduced Temperley-Lieb categories

For singular values of δ_0, the Temperley-Lieb categories give rise to an interesting quotient category. The graph Temperley-Lieb representations supported by the Dynkin quivers factor through this quotient and consequently produce almost Koszul algebra-coalgebra pairs.

5.6.1 Defining the reduced Temperley-Lieb categories

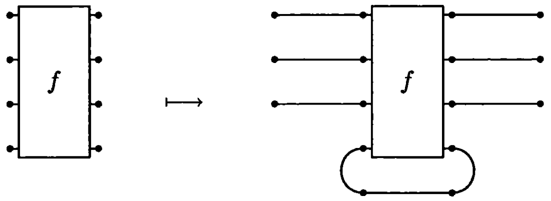
A subcategory $\mathfrak{J}$ of a monoidal category $\mathcal{C}$ is said to be a (TENSOR) IDEAL if it is closed under arbitrary compositions and tensor products, that is,

$$f \in \mathfrak{J} \Rightarrow \begin{cases} hfg \in \mathfrak{J} & \text{for any composable morphisms } g \text{ and } h\,, \\ k \otimes f \otimes l \in \mathfrak{J} & \text{for any morphisms } k \text{ and } l\,. \end{cases}$$

Goodman and Wenzl [GW03] showed that for generic values of the parameter δ, the Temperley-Lieb category contains no proper tensor ideals. However, for singular δ, the Temperley-Lieb category was shown to contain precisely one proper tensor ideal.

Let $\mathfrak{F}$ be a $\mathbb{C}$-linear monoidal functor on $\underline{\mathrm{TL}}$. Then $\mathrm{Ker}(\mathfrak{F})$, the collection of morphisms mapping to a zero morphism under $\mathfrak{F}$, is a tensor ideal. In particular, the kernel of a graph Temperley-Lieb representation is a tensor ideal.

Fix an object $\underline{n} \in \underline{\mathrm{TL}}^{\pm}(\delta_0)$, with $n \geq 1$. There is a $\mathbb{C}$-linear map $p_n\colon \mathrm{End}(\underline{n}) \to \mathrm{End}(\underline{n-1})$ defined by

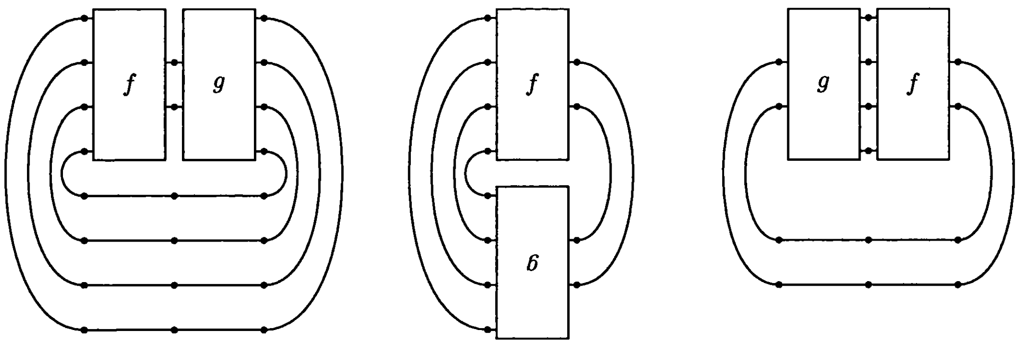


Iterate the above construction to obtain a map $\mathrm{End}(\underline{n}) \to \mathrm{End}(\underline{0})$, for each $\underline{n} \in \underline{\mathrm{TL}}^{\pm}(\delta_0)$, and compose with the identification

$$\begin{aligned} \mathrm{End}(\underline{0}) &\xrightarrow{\sim} \mathbb{C} \\ z\,\mathrm{id}_{\underline{0}} &\longmapsto z\,. \end{aligned}$$

The resulting collection of maps $\mathrm{End}(\underline{n}) \longrightarrow \mathbb{C}$ will be called the TRACE and denoted by $f \mapsto \mathrm{tr}(f)$. It will be convenient on occasion to think of $\mathrm{tr}(f)$ as a morphism in $\underline{\mathrm{TL}}^{\pm}(\delta_0)$ via the above identification $\mathrm{End}(\underline{0}) \cong \mathbb{C}$.

Notice that if f is a diagram from m to n and if g is a diagram from n to m, then $\mathrm{tr}(fg) = \mathrm{tr}(gf)$. This is because each of the diagrams below consists of the same number, r, of closed loops and $\mathrm{tr}(fg) = (\delta_0)^r = \mathrm{tr}(gf)$:



The following two results will play an important role later in the chapter:

Proposition 5.6.1. *Let $\mathfrak{I}$ be a proper tensor ideal in $\underline{TL}^{\pm}(\delta_0)$ and let $f \in \mathfrak{I}$. Then for every composable morphism g,*

$$\mathrm{tr}(fg) = 0\,.$$

Proof. Suppose otherwise. Then $\mathrm{tr}(fg) \in \mathbb{C} \setminus \{0\}$, hence is invertible. But $\mathrm{tr}(fg)$ is built from $fg \in \mathfrak{I}$ by tensor extension and composition with creation and annihilation operators, whence $\mathrm{tr}(fg) \in \mathfrak{I}$. In particular, $\mathfrak{I}$ contains the ideal generated by $1 \in \mathbb{C}$, hence is not proper. □

Write $\delta_0 = [2]_q = q + q^{-1}$ for some $q \in \mathbb{C} \setminus \{0\}$.

Lemma 5.6.2. *The idempotents e_k constructed in Theorem 5.4.3 have trace*

$$\mathrm{tr}(e_k) = [k+1]_q \, .$$

Proof. The idempotent $e_0 \in \mathrm{End}(\underline{0})$ has trace $\mathrm{tr}(e_0) = \mathrm{tr}(\mathrm{id}_{\underline{0}}) = 1$. The proof of Theorem 5.4.3 demonstrates that

$$p_k(e_k) = \frac{[k+1]_q}{[k]_q} \, e_{k-1}$$

and the result now follows. □

Notice that for singular values of q, the idempotent e_{h-1} defining X_{h-1} has zero trace.

The trace induces a symmetric bilinear pairing on opposite Hom-sets defined by

$$\begin{aligned} \langle \cdot , \cdot \rangle \colon \mathrm{Hom}(\underline{m}, \underline{n}) \otimes_{\mathbb{C}} \mathrm{Hom}(\underline{n}, \underline{m}) &\longrightarrow \mathbb{C} \\ f \otimes g &\longmapsto \mathrm{tr}(fg) \, . \end{aligned}$$

Denote by $\mathcal{N}\mathrm{eg}$, the tensor ideal of $\underline{\mathrm{TL}}^{\pm}(\delta_0)$ on which this pairing is degenerate, that is,

$$\mathcal{N}\mathrm{eg}(\underline{m}, \underline{n}) = \{ f \in \mathrm{Hom}(\underline{m}, \underline{n}) \ : \ \mathrm{tr}(fg) = 0 \quad \forall g \in \mathrm{Hom}(\underline{n}, \underline{m}) \} \, .$$

Goodman and Wenzl call this the IDEAL OF NEGLIGIBLE MORPHISMS. They showed that for singular q, the ideal of negligible morphisms is generated by the idempotent e_{h-1} defining X_{h-1}. Furthermore, every proper tensor ideal was shown to contain e_{h-1}, hence coincides with the ideal of negligible morphisms.

Fix a singular value δ_0. The REDUCED TEMPERLEY-LIEB CATEGORY, $\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0)$ is defined to be the categorical quotient of the Temperley-Lieb category $\underline{\mathrm{TL}}^{\pm}(\delta_0)$ by the unique proper tensor ideal. Denote Hom-sets in the reduced Temperley-Lieb category by $\mathrm{Hom}_{\mathrm{red}}(\underline{m}, \underline{n})$. The REDUCED TEMPERLEY-LIEB ALGEBRA, $TL_n^{\mathrm{red}}(\delta_0)$ is the endomorphism algebra $\mathrm{End}_{\mathrm{red}}(\underline{n})$ in the reduced Temperley-Lieb category.

The aim is to show that the analysis performed for the generic Temperley-Lieb category passes to the reduced Temperley-Lieb category. In particular, quotienting by the unique tensor ideal leaves, in an appropriate sense, a semisimple category. Moreover, graph representations of this category can be used to build almost Koszul algebra-coalgebra pairs.

5.6.2 Extending the quotient functor

Use the Yoneda functor to embed the reduced Temperley-Lieb category in the functor category $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0), \underline{\mathrm{Vect}})$. The quotient functor from the Temperley-Lieb category to the reduced Temperley-Lieb category induces a functor between the respective images under the Yoneda embedding:

$$\begin{tikzcd} \underline{\mathrm{TL}}^{\pm}(\delta_0) \arrow[r, "\cong"] \arrow[d, "r"'] & ¥(\underline{\mathrm{TL}}^{\pm}(\delta_0)) \arrow[d, "R"] \\ \underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0) \arrow[r, "\cong"'] & ¥(\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0)) \end{tikzcd} .$$

It is shown that every object in $¥(\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0))$ has a semisimple decomposition in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0), \underline{\mathrm{Vect}})$. This is achieved by extending the functor R to the objects pertinent to the analysis of the Temperley-Lieb categories. Denote Hom-sets in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0), \underline{\mathrm{Vect}})$ by $\mathrm{Hom}_{\mathrm{red}}(\mathfrak{F}, \mathcal{G})$.

Let $A, B \in \underline{\mathrm{TL}}^{\pm}(\delta_0)$ and let V be a vector space. Then R can be extended to $\mathrm{Hom}(\underline{\mathrm{A}} \otimes \mathrm{V}, \underline{B})$ via the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\underline{A} \otimes V, \underline{B}) \arrow[r, "R"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{red}}(\underline{A} \otimes V, \underline{B}) \arrow[d, "\cong"] \\ \mathrm{Hom}(V, \mathrm{Hom}(A, B)) \arrow[r, "r_*"'] & \mathrm{Hom}(V, \mathrm{Hom}_{\mathrm{red}}(A, B)) \end{tikzcd} .$$

Similarly, R has an extension to $\mathrm{Hom}(\underline{B}, \underline{\mathrm{A}} \otimes \mathrm{V})$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\underline{B}, \underline{A} \otimes V) \arrow[r, "R"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{red}}(\underline{B}, \underline{A} \otimes V) \arrow[d, "\cong"] \\
\mathrm{Hom}(B, A) \otimes V \arrow[r, "r \otimes 1"'] & \mathrm{Hom}_{\mathrm{red}}(B, A) \otimes V
\end{tikzcd} .$$

It is also possible to extend R to the image of an idempotent. Suppose R has been defined on W and Z. If $e \in \mathrm{End}(Z)$ is an idempotent then so is $R(e) \in \mathrm{End}_{\mathrm{red}}(Z)$ and R can be extended to $\mathrm{Hom}(W, Z^e)$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(W, Z^e) \arrow[r, "R"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{red}}(W, Z^{R(e)}) \arrow[d, "\cong"] \\
\mathrm{Hom}(W, Z)e \arrow[r, "R"'] & \mathrm{Hom}_{\mathrm{red}}(W, Z)R(e)
\end{tikzcd} .$$

Finally, R can be extended to $\mathrm{Hom}(Z^e, W)$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(Z^e, W) \arrow[r, "R"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{red}}(Z^{R(e)}, W) \arrow[d, "\cong"] \\
e\,\mathrm{Hom}(Z, W) \arrow[r, "R"'] & R(e)\,\mathrm{Hom}_{\mathrm{red}}(Z, W)
\end{tikzcd} .$$

It is simple, if a little tedious, to verify that R is indeed functorial. Let $\underline{m}, \underline{n} \in \underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0)$ and define $\yen(\underline{m}) \otimes \yen(\underline{n}) = \yen(\underline{m} \otimes \underline{n})$ in the reduced functor category $\mathfrak{Fun}^{\circ}(\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}(\delta_0), \underline{\mathrm{Vect}})$. Suppose $e \in \mathrm{End}_{\mathrm{red}}(\yen(\underline{m}))$ and $f \in \mathrm{End}_{\mathrm{red}}(\yen(\underline{n}))$ are idempotents and denote the image functors by $X = \mathrm{Im}(e)$ and $Z = \mathrm{Im}(f)$. Define the tensor product of X and Z to be $X \otimes Z = \mathrm{Im}(e \otimes f)$. Then R is also a monoidal functor.

5.6.3 Analysing the reduced Temperley-Lieb categories

Let $\delta_0 = [2]_q$ for some singular value of $q \in \mathbb{C}\backslash\{0\}$. To simplify notation, henceforth the parameter will be suppressed, thus $\underline{\mathrm{TL}}^{\pm}([2]_q)$ becomes $\underline{\mathrm{TL}}^{\pm}$ and $\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}([2]_q)$ becomes $\underline{\mathrm{TL}}^{\pm}_{\mathrm{red}}$. Define h to be the smallest positive integer such that $[h]_q = 0$. The following lemma is crucial to the subsequent analysis of the reduced Temperley-Lieb category.

Lemma 5.6.3. *Consider the distinguished collection of bricks* $\{X_i : i = 0, \ldots, h-1\}$ *in* $\mathfrak{Fun}^\circ(\underline{TL}^+, \underline{Vect})$ *constructed by Theorem 5.4.3. Then*

1. $R(X_{h-1}) = 0$ *;*

2. $R(X_k)$ *is a brick for* $0 \leq k \leq h-2$ *and* $\mathrm{Hom}_{\mathrm{red}}(R(X_i), R(X_j)) = 0$ *whenever* $i \neq j$.

Proof. The essential ingredient is that the unique proper tensor ideal in $\underline{\mathrm{TL}}^+$ is generated by the idempotent e_{h-1} defining X_{h-1}. Thus, for $\underline{n} \in \underline{\mathrm{TL}}^+$,

$$R(X_{h-1})(\underline{n}) = r(e_{h-1})\,\mathrm{Hom}_{\mathrm{red}}(\underline{h-1}, \underline{n}) = 0\ .$$

For (2), provided $i \neq j$ then $\mathrm{Hom}(X_i, X_j) = 0$, whence the second part. It remains therefore to show that X_k is a brick for $0 \leq k \leq h-2$. But $\mathrm{End}(X_k) \cong \mathbb{C}e_k$ and $\mathrm{tr}(e_k) = [k+1]_q \neq 0$ provided $0 \leq k \leq h-2$. Proposition 5.6.1 ensures that e_k cannot be in the tensor ideal and must therefore have a non-zero image in the quotient category, whence $\mathrm{End}(R(X_k)) \cong \mathbb{C}r(e_k) \cong \mathbb{C}$. □

Identify the functor category $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}^+_{\mathrm{red}}, \underline{\mathrm{Vect}})$ with the subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}^+, \underline{\mathrm{Vect}})$ consisting of the functors that vanish on $\mathcal{N}\mathrm{eg}$. The bricks $R(X_k)$ are thus identified with the quotient of the bricks X_k by a proper subobject, $X_p^{\mathcal{N}\mathrm{eg}}$.

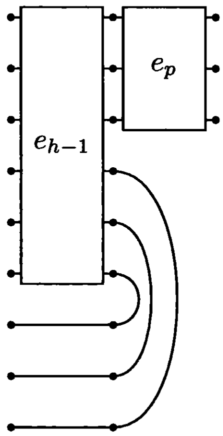
Proposition 5.6.4. *For singular* q *and for* $p \leq h-2$, *the functor* X_p *has a proper subobject* $X_p^{\mathcal{N}\mathrm{eg}}$ *defined by*

$$\begin{aligned} X_p^{\mathcal{N}\mathrm{eg}}(\underline{n}) &= \mathcal{N}\mathrm{eg}(\underline{n}, \underline{p})\, e_p\ , \\ X_p^{\mathcal{N}\mathrm{eg}}(\phi) &= \phi^*\ . \end{aligned}$$

Notice that $X_{h-1}^{\mathcal{N}\mathrm{eg}}$ is precisely X_{h-1} since e_{h-1} generates $\mathcal{N}\mathrm{eg}$.

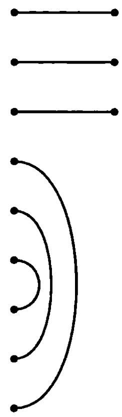
Proof. The functor $X_p^{\mathcal{N}\mathrm{eg}}$ is well-defined since $\mathcal{N}\mathrm{eg}$ is an ideal and is a subobject of X_p via the natural inclusion maps. Furthermore, $e_p \in X_p(\underline{p})$ but e_p is not an element of $\mathcal{N}\mathrm{eg}(\underline{p}, \underline{p})$ because it has a non-zero trace. Hence $e_p \notin X_p^{\mathcal{N}\mathrm{eg}}(\underline{p})$. It remains to show that $X_p^{\mathcal{N}\mathrm{eg}}$ is not the zero functor. Consider the following morphism:

α =



It is an element of $X_p^{\mathcal{N}\mathrm{eg}}(\underline{2(h-1)-p})$ and it remains to show that it is non-zero. It is shown that, in the expression of α as a linear combination of planar Brauer diagrams, the diagram

ψ =



occurs with coefficient 1. First notice that in the definition of α, the e_p can be absorbed by the e_{h-1} and write $e_{h-1} = \mathrm{id}_{\underline{h-1}} + \phi$, where ϕ is an element of $\mathcal{F}_{h-2}(\underline{h-1})$. Thus

$$\alpha \;=\; \psi + (\phi \otimes \mathrm{id}_{\underline{h-1-p}})\psi\,.$$

It remains to show that ψ has coefficient 0 in the expression of $(\phi \otimes \mathrm{id}_{\underline{h-1-p}})\psi$ as a linear combination of planar Brauer diagrams. Let $\gamma \neq \mathrm{id}_{\underline{h-1}}$ be a diagram from $\underline{h-1}$ to $\underline{h-1}$. Then γ induces a pairing on two consecutive points on the left edge. This pairing is invariant under composition on the right, hence the same points remain paired in the diagram $(\gamma \otimes \mathrm{id}_{\underline{h-1-p}})\psi$. This diagram cannot be a multiple of ψ, whence the result. $\square$

Corollary 5.6.5. *Let q be singular and let $p \leq h-2$. Extend $R(X_p)$ to a functor*

$\underline{TL}^+ \to \underline{Vect}$ *that vanishes on* $\mathcal{N}$eg. *Then* $R(X_p)$ *is the cokernel of the inclusion map* $X_p^{\mathcal{N}\mathrm{eg}} \to X_p$.

Proof. Cokernels in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}^+, \underline{\mathrm{Vect}})$ are defined pointwise. Now

$$R(X_p)(\underline{n}) = \frac{\mathrm{Hom}(\underline{n}, \underline{p})}{\mathcal{N}\mathrm{eg}(\underline{n}, \underline{p})} \cdot R(e_p) = \frac{\mathrm{Hom}(\underline{n}, \underline{p}) e_p}{\mathcal{N}\mathrm{eg}(\underline{n}, \underline{p}) e_p} = \frac{X_p(\underline{n})}{X_p^{\mathcal{N}\mathrm{eg}}(\underline{n})} ,$$

as required. □

The functor R is additive and monoidal, hence the bricks $R(X_p)$ satisfy a truncated fusion rule:

$$\begin{aligned} R(X_p) \otimes R(X_1) &\cong R(X_{p-1}) \oplus R(X_{p+1}) , \qquad \text{for } 1 \le p \le h-3 , \\ R(X_0) \otimes R(X_1) &\cong R(X_1) , \\ R(X_{h-2}) \otimes R(X_1) &\cong R(X_{h-3}) . \end{aligned}$$

In particular, a simple induction proves that $\{R(X_p) : 0 \le p \le h-2\}$ is a complete set of indecomposable summands for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}^+_{\mathrm{red}}, \underline{\mathrm{Vect}})$ generated by $¥(\underline{\mathrm{TL}}^+_{\mathrm{red}})$.

Subsequent analysis of the reduced Temperley-Lieb category can now proceed in an identical fashion to that for the Temperley-Lieb category for generic values of q. In particular,

Theorem 5.6.6. *The functors* $R(X_p)$ *are simple in* $\mathfrak{Fun}^\circ(\underline{TL}^+_{\mathrm{red}}, \underline{Vect})$.

Corollary 5.6.7. *The abelian subcategory of* $\mathfrak{Fun}^\circ(\underline{TL}^\pm_{\mathrm{red}}, \underline{Vect})$ *generated by* $¥(\underline{TL}^\pm_{\mathrm{red}})$ *is semisimple.*

Theorem 5.6.8. *For each* n*, the* TL_n^{red}*-module* $\mathrm{Hom}_{\mathrm{red}}(R(X_p), ¥(\underline{n}))$ *is simple.*

Theorem 5.6.9. *For every* $\underline{n} \in \underline{TL}^+_{\mathrm{red}}$*, the object* $¥(\underline{n})$ *is semisimple as a* TL_n^{red}*-module and has a canonical direct sum decomposition*

$$¥(\underline{n}) \cong \bigoplus_{p=0}^{h-2} R(X_p) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{red}}(R(X_p), ¥(\underline{n})) .$$

5.6.4 Almost Koszul pairs from reduced representations

A REDUCED GRAPH TEMPERLEY-LIEB REPRESENTATION is a graph Temperley-Lieb representation factoring through the unique proper tensor ideal.

Proposition 5.6.10. *A Dynkin quiver together with a symplectic form defines a reduced graph Temperley-Lieb representation.*

Proof. The representation defined in Section 5.1.2 is certainly non-trivial so it remains to show that the kernel is also non-trivial. Extend the representation as in Section 5.5.2. Denote by M_i the decomposition matrix (Section 2.3.1) of the $(\mathbb{C}Q)_0$-bimodule $\mathfrak{F}(X_i)$. Then the matrices M_i satisfy the fusion rule

$$\begin{aligned} M_i M_1 &= M_{i+1} + M_{i-1} \,, \qquad \text{for } 1 \leq i \leq h-2 \,, \\ M_0 M_1 &= M_1 \,. \end{aligned}$$

Moreover, $M_0 = I$ and M_1 is the adjacency matrix of the Dynkin quiver. This is an initial segment of the SL_2 matrix recurrence and Proposition 3.3.4 established that $M_{h-1} = 0$. Hence, $\mathfrak{F}(X_{h-1})$ is trivial and the idempotent $\mathfrak{F}(e_{h-1})$ must be a zero morphism. The kernel of the graph Temperley-Lieb representation must therefore coincide with the unique proper tensor ideal in the Temperley-Lieb category. □

Extend the reduced graph Temperley-Lieb representation $\mathfrak{F}$ as in Section 5.5.2. Then for each n, the TL_n^{red}-module $\mathfrak{F}(\underline{n})$ has a canonical semisimple decomposition

$$\mathfrak{F}(\underline{n}) \cong \bigoplus_{p=0}^{h-2} \mathfrak{F}(X_p) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{red}}(R(X_p), \yen(\underline{n})) \,. \tag{5.16}$$

To conclude the chapter, it is shown that reduced graph Temperley-Lieb representations define an almost Koszul algebra-coalgebra pair.

Recall that for singular q, the functors X_p with $p \geq h$ are defined to be the zero functor. Define

$$\begin{aligned} \widetilde{X} &= \bigoplus_p R(X_p) \,, \\ \widetilde{A} &= R(X_0) \oplus R(X_1) \oplus R(X_0) \,, \end{aligned}$$

and equip $\widetilde{X}$ and $\widetilde{A}$ with the induced algebra and coalgebra structures from X and A respectively.

Theorem 5.6.11. *The formal algebra-coalgebra pair $(\widetilde{X}, \widetilde{A})$ is almost Koszul.*

Proof. Theorem 5.4.9 and the fact that R is additive and monoidal ensure that, for every $2 \leq k \leq h-1$, the following short sequences are exact,

$$0 \longrightarrow \widetilde{X}_{k-2} \otimes \widetilde{A}_2 \longrightarrow \widetilde{X}_{k-1} \otimes \widetilde{A}_1 \longrightarrow \widetilde{X}_k \longrightarrow 0 ,$$

where the maps are given by the Koszul differential on $\widetilde{X}_\bullet \otimes \widetilde{A}_\bullet$. In particular, $\widetilde{X}_{h-1} = R(X_{h-1}) = 0$. Thus the short exact sequence for $k = h-1$ degenerates to give $\widetilde{X}_{h-3} \otimes \widetilde{A}_2 \cong \widetilde{X}_{h-2} \otimes \widetilde{A}_1$ as required. □

Let $\mathfrak{F}$ be a reduced graph Temperley-Lieb representation and recall the definitions from Section 5.5.3:

$$\begin{aligned} \Sigma &= \bigoplus_p \mathfrak{F}(X_p) , \\ \Lambda &= \mathfrak{F}(X_0) \oplus \mathfrak{F}(X_1) \oplus \mathfrak{F}(X_0) , \\ \Pi &= \frac{\mathbb{C}Q}{\mathfrak{I}} , \text{ where } \mathfrak{I} \text{ is the two-sided ideal in } \mathbb{C}Q \text{ generated by } \mathrm{Im}(\mathfrak{C}). \end{aligned}$$

Since $\mathfrak{F}$ is a reduced representation, there is an immediate corollary to Theorem 5.6.11:

Corollary 5.6.12. *The algebra-coalgebra pair* (Σ, Λ) *is almost Koszul.*

Corollary 5.5.9 assures that $\Sigma \cong \Pi^{(\leq h-1)}$. However, $\mathfrak{F}$ is a reduced representation, hence $\Pi_{h-1} \cong \Sigma_{h-1} = \mathfrak{F}(X_{h-1}) = 0$. But Π is generated by Π_1, so $\Pi_k = 0$ whenever $k \geq h-1$. Thus $\Pi = \Pi^{(\leq h-1)}$. Consequently:

Proposition 5.6.13. *The algebras* Σ *and* Π *are isomorphic.*

Corollary 5.6.14. *The algebra-coalgebra pair* (Π, Λ) *is almost Koszul.*

Remark 5.6.15. *Equivalently, this follows from the observation that, since the coalgebra is bounded in degree* ≥ 2*, an almost Koszul algebra is quadratic ([BBK02]).*

In particular, recall that the preprojective algebra of a Dynkin quiver with a symplectic form ω coincides precisely with the algebra $\Pi = \frac{\mathbb{C}Q}{\mathrm{Im}(\mathfrak{C})}$ for the action defined in Section 5.1.2. The result of Brenner, Butler and King [BBK02] is now recovered:

Corollary 5.6.16. *Let* (Q, ω) *be a Dynkin quiver together with a symplectic form and denote the preprojective algebra by* Π*. Then the pair* (Π, Λ) *is almost Koszul.*

Notes

The Temperley-Lieb algebras (or more strictly, representations of these algebras) first appeared in a paper by Temperley and Lieb [TL71] studying the 6 vertex (ice-type) model and have since become an integral part of transfer matrix methods for many models in statistical mechanics. Vaughan Jones [Jon83] realised the importance of the Temperley-Lieb algebras in studying the inclusions of Von Neumann algebras ([GdlHJ89] is a good starting point) and exploited its relation to the braid groups to define a polynomial invariant of knots [Jon85]. Brauer diagrams were known to Weyl [Wey39] and first used in connection with the Temperley-Lieb algebras by Kauffman [Kau87] who used the diagram calculus to define a state model for the Jones polynomial.

This widespread interest in the Temperley-Lieb algebras ensures that they have been extensively studied. A through account of their properties and connection to statistical mechanics is given in [Mar91] and I would also recommend [Wes95] as an introduction to the representation theory of the Temperley-Lieb algebras. The idempotents constructed by Theorem 5.4.3 have been "rediscovered" on numerous occasions but are generally attributed to Jones and Wenzl ([Wen87, Wen88]).

The representation of the Temperley-Lieb algebras obtained by using a symmetric (symplectic) quiver and a Perron-Frobenius eigenvector appears in [Pas87] (modified to make the operators self-adjoint for the 'standard' inner product on $(\mathbb{C}Q)_n$) and with a non-vanishing eigenvector in [DFZ90a, DF92]. That these representations are unitary and factor through the reduced Temperley-Lieb algebras was well known, but that this amounts to quotienting the Temperley-Lieb category by the unique tensor ideal was conjectured by Freedman [Fre03] and proved by Goodman and Wenzl [GW03].

The Temperley-Lieb category appears in [Tur94], which also details Turaev's approach to completing the category. Formal idempotent completion appears as an exercise in [Fre64], but I prefer the embedding into the functor category, which led to some interesting statements about 'modules for the Temperley-Lieb category'. The Yoneda embedding and the importance of functor categories as a tool for enriching and studying the original category is well known (see [Fre64]).

Chapter 6

Rational conformal field theory

This chapter provides a connection between the analysis of the previous chapter and certain aspects of rational conformal field theory.

6.1 Segal's category

There are a number of different approaches to conformal field theory. The story that unfolds in this chapter will touch upon three of these and some of the connections between them.

One of the most influential and enduring approaches to conformal field theory is due to Graeme Segal. According to Segal [Seg89] (see also [Fio06]), a conformal field theory is a functor from a category $\underline{\text{Segal}}$ to $\underline{\text{Vect}}$, satisfying various properties. The principal ideas are as follows.

Objects of $\underline{\text{Segal}}$ are disjoint unions of parameterised circles and morphisms between them are cobordisms given by Riemannian surfaces up to conformal equivalence. The domain and codomain of the morphism are distinguished by the orientation of the circles and are called *in-boundaries* and *out-boundaries* respectively. An example of a (cobordism representing a) morphism in $\underline{\text{Segal}}$ is given in Figure 6-1 below.

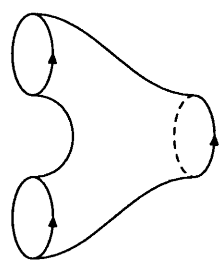
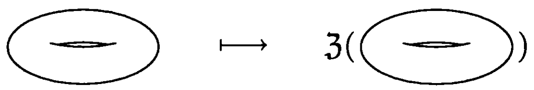


Figure 6-1: A morphism in Segal from two circles to one circle.

The category Segal has a tensor product defined by disjoint union. A conformal field theory is, in particular, a (strict) monoidal functor $\mathfrak{Z}$: Segal $\to$ Vect. Hence, a conformal field theory assigns a vector space W to each circle and a linear map $W^{\otimes m} \longrightarrow W^{\otimes n}$ to every equivalence class of cobordisms from m circles to n circles.

A conformal field theory will, in particular, assign to a torus a complex number depending only on the conformal equivalence class of the torus, that is, depending on its modulus τ:



Thus part of the information of a conformal field theory is a *modular invariant function.*

6.2 Modular invariants and the Virasoro algebra

Belavin, Polyakov and Zamolodchikov ([BPZ84, MS89]) prefer to define a conformal field theory as a bimodule for the Virasoro algebra (or possibly an extension of it), together with some additional information. Schottenloher [Sch97] provides a good introduction to the Virasoro algebra and conformal field theory.

The VIRASORO ALGEBRA, $\mathcal{V}$, is generated by elements $\{L_n \ : \ n \in \mathbb{Z}\}$ and a central element C, subject to the following commutation relations (for $n, m \in \mathbb{Z}$):

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \delta_{n+m,0}\frac{n}{12}(n^2-1)C \,, \\
[L_n, Z] &= 0 \,.
\end{aligned}$$

Fix an irreducible representation of $\mathcal{V}$. Then the central element C acts as multiplication by a scalar c, called the central charge of the representation. A conformal field theory is said to be RATIONAL if it is a direct sum of finitely many irreducible Virasoro bimodules and if the central element C always acts as multiplication by a rational number. Henceforth, suppose that a value c by which the central element acts has been fixed.

The Virasoro algebra is the unique proper central extension of the Witt algebra, which has a representation as the complex-valued polynomial vector fields on the circle. It can be seen (or at least, intuited) how an action of the Virasoro algebra might arise in Segal's approach.

In particular, the modular invariant function for the torus with modular parameter $g = \exp(2\pi i\tau)$ is expected to have a natural decomposition

$$\mathfrak{Z}(\tau) = \sum_{i,j} N_{i,j}\chi_i(g)\overline{\chi_j}(g) , \tag{6.1}$$

where χ_i is a character of an irreducible Virasoro module on which the central element C operates as multiplication by c.

Capelli, Itzykson and Zuber [CIZ87] achieved the first classification of (*unitary*) rational conformal field theories for central charges $c < 1$, by identifying all possible modular invariant functions of the form (6.1). They observed that the permissible modular invariant functions are in one-to-one correspondence with the ADE Dynkin diagrams. In particular, the characters χ_i are naturally labelled by $i \in \{0, \ldots, h-2\}$, where h is the Coxeter number of the appropriate Dynkin graph and the diagonal terms in the multiplicity matrix N encode the Coxeter exponents of the diagram:

$$N_{i,i} = \begin{cases} 1 & \text{if } i \text{ is a Coxeter exponent,} \\ 0 & \text{otherwise.} \end{cases}$$

6.3 Boundary conformal field theory and fusion rules

It is a remarkable fact, and by no means clear in the present formulation, that the characters appearing in the decomposition formula for $\mathfrak{Z}$ form a basis for a finite-dimensional representation of the double cover of the modular group ([Zub02]).

Consider the modular transformation $\tau \mapsto -\frac{1}{\tau}$. Let I denote an index set for the characters appearing in the decomposition of Z and define S to be the matrix representing the induced action on the characters. Thus, if $g = \exp(2\pi i\tau)$ and $g' = \exp(\frac{-2\pi i}{\tau})$ then

$$\chi_j(g') = \sum_{i \in I} S^i_j \chi_i(g) .$$

The characters χ_i possess a particular ring structure (the *fusion product* $\star$) induced by the operator product algebra of Quantum Field Theory ([Zub02]). Verlinde [Ver88] realised that he could relate the matrix entries S_{ij} to these fusion rules for the characters. In the $c < 1$ case, the characters χ_i are in a natural one-to-one correspondence with a finite segment of the representations of SL_2 and satisfy a truncated SL_2 fusion rule:

$$\chi_i \star \chi_1 = \chi_{i-1} + \chi_{i+1} ,$$

where $i \in \{0, \ldots, h-2\}$ and the character χ_j is zero by convention if $j \in \{-1, h-1\}$. Let r^k_{ij} denote the multiplicity of χ_k in the product $\chi_i \chi_j$, that is,

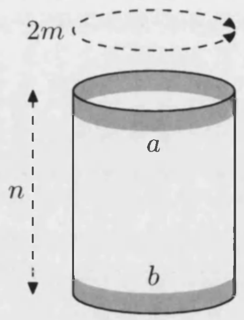
$$\chi_i \star \chi_j = \sum_{k=0}^{h-2} r^k_{ij} \chi_k .$$

Verlinde's formula states that

$$\sum_k S^p_k r^k_{ij} = \frac{S^p_i S^p_j}{S^p_0} .$$

Cardy [Car89] sought to give physical meaning to the fusion rules and one-sided representations of the Virasoro algebra that arise in Verlinde's calculations. He argued that *boundary* conformal field theories are the appropriate objects to study. In Segal's formalism, the morphisms should be replaced by *cobordisms with boundary*, thus distinguishing two types of boundary: the in-boundaries and out-boundaries determined by the objects and additional boundaries that must be labelled by *conformally invariant boundary conditions.*

In place of the torus, the principal object of study is a cylinder, to which a boundary conformal field theory must associate a number $Z_{a,b}(g)$, depending only on the boundary conditions a and b at the top and bottom of the cylinder and the conformal parameter $g = \exp\left(\frac{2\pi i m}{n}\right)$, where $2m$ is the circumference and n is the length of the cylinder (see Figure 6-2 below).



$$\longmapsto \qquad Z_{a,b}\left(\exp\left(\tfrac{2\pi i m}{n}\right)\right)$$

Figure 6-2: The *partition function* of a cylinder with boundary conditions a and b.

Cardy details how an action of the Virasoro algebra arises and deduces that $Z_{a,b}$, the *partition function* of the cylinder, has an expansion in terms of characters for the Virasoro algebra of the form

$$Z_{a,b}(g) = \sum_{i \in I} M_i(a,b)\chi_i(g) \ .$$

A boundary conformal field theory is RATIONAL if the index set I is finite.

For the $c < 1$ rational unitary boundary conformal field theories, there exists a conformally invariant boundary state (a, b) corresponding to each pair of vertices a and b on an appropriate ADE Dynkin graph Q. The multiplicities M_i are thus matrices, whose entries are labelled by pairs of vertices. Let h denote the Coxeter number of Q. The matrices M_i are further constrained to satisfy the truncated SL_2 fusion rules:

$$\begin{aligned} M_i M_1 &= M_{i-1} + M_{i+1} \ , \quad 0 \leq i \leq h-2 \ , \\ M_0 &= I \ , \\ M_{-1} &= M_{h-1} = 0 \ , \end{aligned}$$

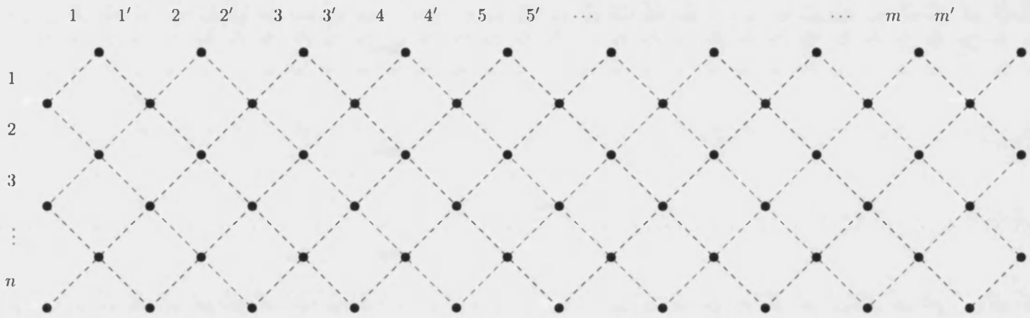
yielding a representation of the ring of characters, or NIM-Rep in the physics literature (a non-negative integer matrix representation).

NIM-Reps help clarify the connection between conformal field theories and quivers (a non-negative integer matrix is the adjacency matrix for a quiver) and have proved a key tool in the classification of rational conformal field theories for values of c greater than 1.

6.4 Lattice models

Conformal field theories also arise as a continuum limit of lattice models that exhibit a second-order phase transition. Pasquier [Pas87] realised that the existing *restricted solid-on-solid* models of Andrews, Baxter and Forrester [ABF84] correspond to the lattice models for the A_n series of conformal field theories and he successfully constructed a generalisation of these models that depend explicitly on the Dynkin graphs. The Pasquier model for a cylinder is described below.

Consider a diamond lattice as in the diagram below, with edges labelled by pairs (i, j) or (i, j'), where $1 \leq i \leq n$ and $j \in \mathbb{N}$.



Wrap the lattice on a cylinder, by identifying the edge $(i, m+1)$ with the edge $(i, 1)$ for each i and denote the resulting lattice by L.

Fix a Dynkin graph Q. A STATE (or CONFIGURATION) on L is a morphism of graphs $L \to Q$. Thus, each vertex of L is labelled by a vertex of Q with the constraint that neighbouring vertices on the lattice L must be labelled by neighbouring vertices on the graph Q. Each edge of L is labelled by an edge of Q connecting the corresponding vertices.

Choose two nodes a and b on the Dynkin quiver Q as boundary conditions for the model. Henceforth, only states whose restriction to each vertex in the top boundary is a and the bottom boundary is b will be considered. Thus a state can be thought of as a family of small deformations of a path of length n from a to b on the quiver Q. Denote the set of permissible states by Σ.

The primary object of interest in the model is the PARTITION FUNCTION

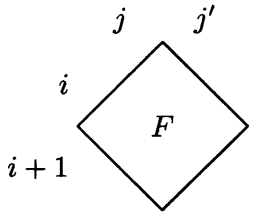
$$Z_{a,b}(x) = \sum_{\sigma} B_{\sigma}(x) \; ,$$

where B is an *energy* function

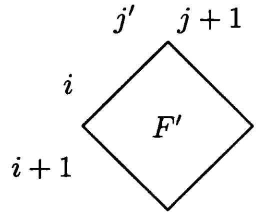
$$\begin{aligned} B \colon \Sigma &\longrightarrow C^{\infty}(\mathbb{R}, \mathbb{R}) \\ \sigma &\longmapsto (x \mapsto (B_{\sigma}(x))) \; . \end{aligned}$$

One of the key techniques for computing the partition function of a model is the *transfer matrix.* These also provide a convenient way of presenting the energy function B in the Pasquier model.

The overall energy of a state $B_{\sigma}(x)$ is a product of local energies (*face weights*) $B_{\sigma}^{F}(x)$ assigned to (the restriction of σ to) each face F of the diamond lattice. There are two types of diamond face in the lattice L, as distinguished by the notation for the edges:



and



In the transfer model formalism, the face weights $B_{\bullet}^{F}(x)$ are considered as $(\mathbb{C}Q)_0$-bilinear functions $(\mathbb{C}Q)_2 \to (\mathbb{C}Q)_2$. Thus $B_{\bullet}^{F}(x)$ maps a path of length two on Q to a linear combination of paths of length two on Q whilst preserving the endpoints of the path.

Statistical mechanical models with second order phase transitions may be built using the Temperley-Lieb algebra. Pasquier defined therefore an operator $U \colon (\mathbb{C}Q)_2 \to (\mathbb{C}Q)_2$, an action of the basic Temperley-Lieb operator $\supset\subset$, that extends by tensor extension to a representation of TL_n on $(\mathbb{C}Q)_n$. The face transfer operators are now defined by

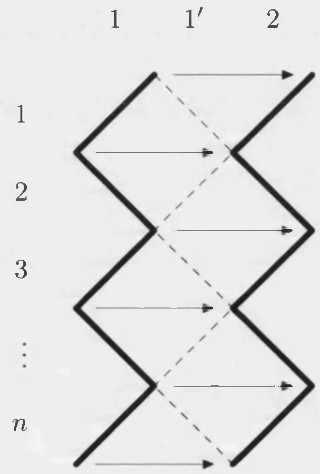
$$B_{\bullet}^{F}(x) = \mathrm{id} + xU \qquad \text{and} \qquad B_{\bullet}^{F'}(x) = x\,\mathrm{id} + U \; .$$

On the boundary faces, define the transfer matrix to be (a multiple of) the identity operator $(\mathbb{C}Q)_1 \to (\mathbb{C}Q)_1$.

Notice that (up to scale) the face transfer operators are deformations of the braid operators $\mathrm{id} - q^{\pm 1} \supset\subset$ in the Temperley-Lieb category. A useful mental picture is that of the face transfer operators 'weaving' the cylinder as in Figure 6-3 below.

Figure 6-3: 'Weaving' the cylinder.

The *column-to-column* transfer operator is a product of two consecutive columns of face transfer operators



and thus has the expression

$$T(x) = \left(\prod_{i \text{ odd}} (\mathrm{id} + xU_i) \right) \left(\prod_{i \text{ even}} (x\,\mathrm{id} + U_i) \right) ,$$

where, for $1 \leq i \leq n-1$, the tensor extension U_i is defined by

$$U_i = \overbrace{1 \otimes \cdots \otimes 1}^{i-1 \text{ times}} \otimes U \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-1-i \text{ times}} .$$

A simple calculation now shows that the partition function Z can be expressed as

$$Z_{a,b}(x) = \mathrm{tr}(T^m) .$$

Pasquier's choice of action, U, is equivalent to that constructed in Section 5.1.2 with the aid of a symplectic form on the Dynkin graph and the Perron-Frobenius eigenvector. In particular, the operator T represents an element of the *reduced* Temperley-Lieb algebra acting on $(\mathbb{C}Q)_n$. Hence, by Theorem 5.6.9, Z decomposes as

$$Z_{a,b}(x) = \sum_{0 \leq p \leq h-2} M_p(a,b)\, \mathrm{tr}(T^m|_{\mathrm{Hom}_{\mathrm{red}}(R(X_p),\mathfrak{Y}(\underline{n}))}) ,$$

where M_p is the $(\mathbb{C}Q)_0$-decomposition matrix for the reduced brick $R(X_p)$. Notice that the fact that the sum is finite (the lattice model is *rational*) is precisely because the TL_n-action chosen extends to a *reduced* representation of the Temperley-Lieb category.

The boundary conformal field theory should now arise as a continuum limit of the lattice model, taking m and n to infinity whilst keeping the ratio $\frac{2m}{n}$ constant and letting x approach a critical value at an appropriate speed. In particular, the decomposition formula for the partition function of the cylinder arises at the discrete level, with the reduced Temperley-Lieb algebras playing the role of the Virasoro algebra ([Dor93]).

Precisely what happens in the limit, or even in what sense a family of representations for the Temperley-Lieb algebras should become a representation of the Virasoro algebra is not well-understood. Moreover, the lattice model in principle contains a vast deal more information than the conformal field theory alone and has other interesting limits from the physical point of view.

6.5 The Temperley-Lieb category as a universal lattice model

From the category theory point of view, a limit in a category $\mathcal{C}$ should be defined as the (co)limit of a functor $\mathfrak{F}\colon \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is an index category.

In the case of the Pasquier models the approach of the previous chapter provides a natural setting for trying to understand the limit. Graph Temperley-Lieb representations are functors $\underline{\mathrm{TL}} \to (\mathbb{C}Q)_0$-$\underline{\mathrm{mod}}$-$(\mathbb{C}Q)_0$ and encode all the information in the corresponding Pasquier model. This suggests that a natural place to start looking for the appropriate limit is the category of $(\mathbb{C}Q)_0$-bimodules with the (reduced) Temperley-Lieb category (or an appropriate subcategory) as the indexing category.

More importantly, a graph Temperley-Lieb representation contains significantly more information than any of its (co)limits. Thus it seems natural and possibly fruitful, to study the Temperley-Lieb category and graph Temperley-Lieb representations if one wants to understand the corresponding lattice model and any one of its limits.

6.6 Towards SL_3

The Temperley-Lieb category and its representations are useful objects of study in order to understand the $c < 1$ boundary conformal field theories. For the $1 < c < 2$ theories a different category will need to be considered.

The Temperley-Lieb category is the tensor subcategory of $\mathcal{U}_q[\mathfrak{sl}_2]$-$\underline{\mathrm{mod}}$ generated by the fundamental representation V of the quantum group $\mathcal{U}_q[\mathfrak{sl}_2]$. This, in particular, is what gives rise to the SL_2 fusion rule $M_p M_1 = M_{p-1} + M_{p+1}$ for the multiplicity matrices.

For the $1 < c < 2$ boundary conformal field theories, Di Francesco and Zuber [DFZ90a] demanded that the multiplicities $M_{i,j}$ obey the (truncated) SL_3 fusion rules:

$$\begin{aligned} M_{i,j}M_{1,0} &= M_{i,j-1} + M_{i-1,j+1} + M_{i+1,j} \ , \\ M_{i,j}M_{0,1} &= M_{i-1,j} + M_{i+1,j-1} + M_{i,j+1} \ , \end{aligned}$$

with

$$\begin{aligned} M_{1,0} &= (M_{0,1})^* \ , \\ M_{1,0}M_{0,1} &= M_{0,1}M_{1,0} \ , \\ M_{0,0} &= I \end{aligned}$$

and the convention is taken that if either of the indices are negative then the corresponding matrix is zero.

Thus it would appear that the appropriate object to study would be the tensor subcategory of $\mathcal{U}_q[\mathfrak{sl}_3]$-$\underline{\mathrm{mod}}$ generated by the fundamental representations V and V^* of the quantum group $\mathcal{U}_q[\mathfrak{sl}_3]$.

The Pasquier model needs to be modified slightly: a column in the lattice can still correspond to a path of length n on some quiver Q (but notice that Q is no longer necessarily symmetric) and the column-to-column transfer matrix will be an element of $\mathrm{End}_{\mathcal{U}_q[\mathfrak{sl}_3]\text{-}\underline{\mathrm{mod}}}(V^{\otimes n})$, a different quotient of the *Hecke algebra.*

The classification of rational boundary conformal field theories appears to have, more or less, continued in this way. Zuber and Di Francesco have led the search for quivers whose adjacency matrices satisfy the truncated SL_3 fusion rules (NIM-Reps) and that support an action of the Hecke algebras. This approach culminated in the publication [DFZ90b] of a list of quivers for the $1 < c < 2$ rational boundary conformal field theories (see Figures 3-4 and 3-5). Ocneanu, by his own methods, independently constructed a list of quivers for the $1 < c < 2$ theories and explained why one of the quivers in the Zuber-Di Francesco list had to be removed ([Zub02]). Ocneanu [Ocn02] has since published lists corresponding to the SL_4 fusion rules. It appears that Ocneanu's methods are closely related to the functorial point of view taken in this thesis. Circumstantial evidence, in the form of consistency equations necessary for the existence of graph representations appear in Appendix B (see also [CS06]).

On the modular invariant side, Gannon has published lists of the permissible modular invariants for SU(3) ([Gan94]) and for all $\mathrm{SU}(n)_k$ where the cutoff k occurs at level 1, 2 and 3 ([Gan97]). Unlike the $n = 2$ case, these modular invariants are not in a one-to-one correspondence with the quivers arising in the NIM-rep classification ([Zub02, Gan02]).

The next chapter is devoted to a study of a category, $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$, that should be the tensor subcategory of $\mathcal{U}_q[\mathfrak{sl}_3]$-$\underline{\mathrm{mod}}$ generated by the fundamental representations V and V^* of the quantum group $\mathcal{U}_q[\mathfrak{sl}_3]$. The primary motivation for this is to exhibit another universal algebra-coalgebra pair, representations of which should provide new and interesting examples of almost Koszul pairs. It is hoped that the categorical approach pursued might also be of interest to researchers in conformal field theory (and other disciplines) with varying degrees of familiarity with the results obtained.

Chapter 7

The diagram category, $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$

The analysis of the Temperley-Lieb category is mimicked for the category $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$, which is an $\mathfrak{sl}_3$ analogue of $\underline{\mathrm{TL}}^-$. For generic q, graph representations of this category produce Koszul algebra-coalgebra pairs. For singular q, graph representations factoring through a certain tensor ideal produce almost Koszul algebra-coalgebra pairs.

7.1 Defining the diagram category

In this section, the diagram category $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ will be introduced and some elementary properties established.

7.1.1 Objects

The objects of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ are finite strings (possibly empty) of the symbols $\bullet$ and $\circ$. A string will be represented pictorially by a column of black and white dots, with the first string entry at the head of the column and the last string entry at the foot of the column. There is a tensor product on objects, given by concatenation of strings. The length of a string s is the number of entries in the string and is denoted by $|s|$. There is an operation called CONJUGATION on objects defined by reversing the colour of each entry in the string and denoted by $\overline{A}$ for an object A. There is an operation called REVERSAL on strings given by reversing the order of the entries in a string and denoted by A' for an object A. Notice that conjugation and reversal commute: the composite operator shall be called DUALITY and denoted by A^* for an object A.

It will occasionally be convenient to use the notation $\underline{1}$ in place of $\bullet$ and $\underline{n} = \underline{1}^{\otimes n}$ to denote the string consisting of precisely n black dots. The notation $\underline{1}^*$ will be preferred in designating a white dot and the empty string will be denoted by $\underline{0}$.

7.1.2 Generalised diagrams

It is convenient to begin by describing a larger collection of diagrams before proceeding to specify the morphisms in $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. Let A and B be objects in $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. A generalised diagram from A to B consists of the following:

1. a closed rectangle R in the plane with two opposite edges designated as *left* and *right*;

2. $|A|$ marked points on the left edge labelled top-down by the entries of A and $|B|$ marked points on the right edge labelled top-down by the entries of B;

3. A finite bipartite (directed) graph embedded in the rectangle R for which the marked points on the left and right edges are univalent vertices. The graph must be directed in such a way that black vertices on the left edge are sources and white vertices on the left edge are sinks with the opposite directions prescribed for the right edge. Edges of the embedded graph may only intersect at a vertex and the graph may only intersect the boundary of the rectangle at the $|A| + |B|$ marked univalent vertices.

Two generalised diagrams are considered equivalent if they are isotopic. As in the Temperley-Lieb category, the rectangle is usually omitted. An example of such a generalised diagram is given in Figure 7-1 below.

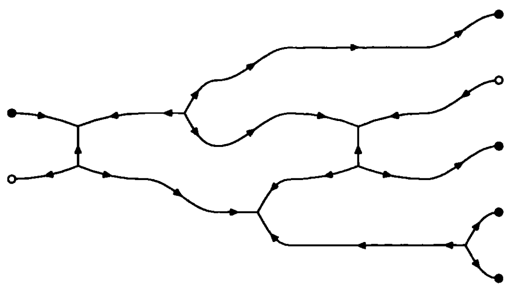


Figure 7-1: A generalised diagram $\underline{1} \otimes \underline{1}^* \to \underline{1} \otimes \underline{1}^* \otimes \underline{3}$.

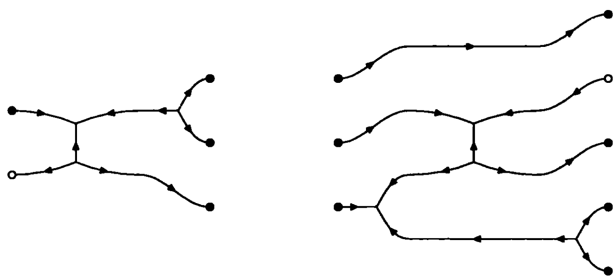
By convention, the rectangle containing the empty graph is a generalised diagram from $\underline{0}$ to $\underline{0}$.

Let f be a generalised diagram from A to B and let g be a generalised diagram from B to C. The composite fg is a generalised diagram from A to C and is defined as follows:

1. Juxtapose the rectangles of f and g, identifying the right edge of f (with its $|B|$ marked points) with the left edge of g (with its $|B|$ marked points).

2. The resulting graph (with the vertices corresponding to the $|B|$ marked points removed) is a generalised diagram fg from A to C.

Notice that, if the vertices corresponding to the string B had been allowed to remain, then fg would not be bipartite. As an example, the generalised diagram in Figure 7-1 is the composite of the following two generalised diagrams



The tensor product on objects extends to generalised diagrams by defining, for $f\colon A \to B$ and $g\colon C \to D$, the tensor product $f \otimes g$ to be the generalised diagram obtained by stacking f and g vertically:

$$f \otimes g = \begin{matrix} f \\ g \end{matrix} : A \otimes C \to B \otimes D \, .$$

It is clear that the identity maps for the composition are tensor products of the generalised diagrams

 $: \underline{1} \to \underline{1}\,,$ $: \underline{1}^* \to \underline{1}^*\,.$

Any generalised diagram of the form $\mathrm{id}_A \otimes f \otimes \mathrm{id}_B$ is said to be a TENSOR EXTENSION of f.

Conjugation on objects extends to generalised diagrams by reversing all the arrows in the generalised diagram and the colours of the boundary vertices. Duality extends by rotating the generalised diagram by π and changing the colours of the boundary vertices. The opposite of a generalised diagram A is defined to be the generalised diagram A° obtained by reflecting A in an axis parallel to the left and right edges of the rectangle and reversing the direction of every arrow.

7.1.3 Morphisms

Consider the following distinguished set of generalised diagrams:

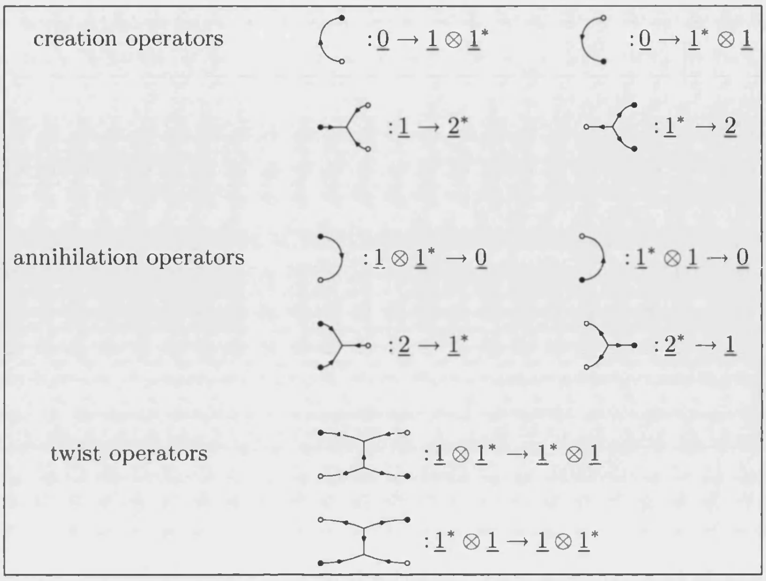
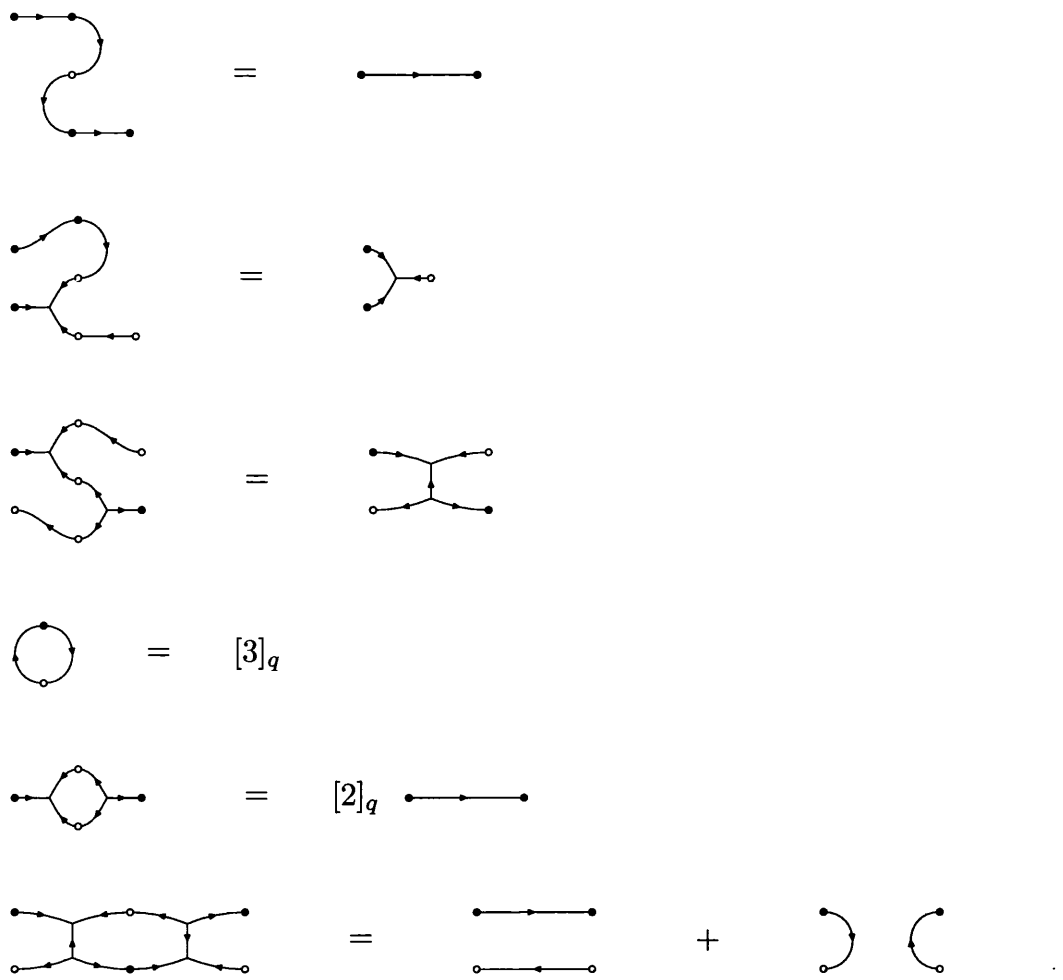


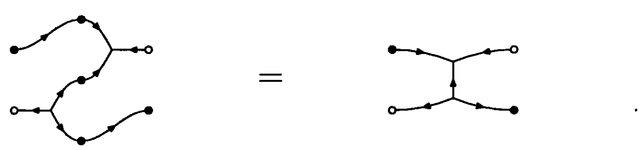
Figure 7-2: A list of generators.

Let $\mathcal{D}$ denote the set of generalised diagrams obtained by composing tensor extensions of the generalised diagrams in Figure 7-2 above (together with the identity diagrams). Elements of $\mathcal{D}$ will be called simply DIAGRAMS.

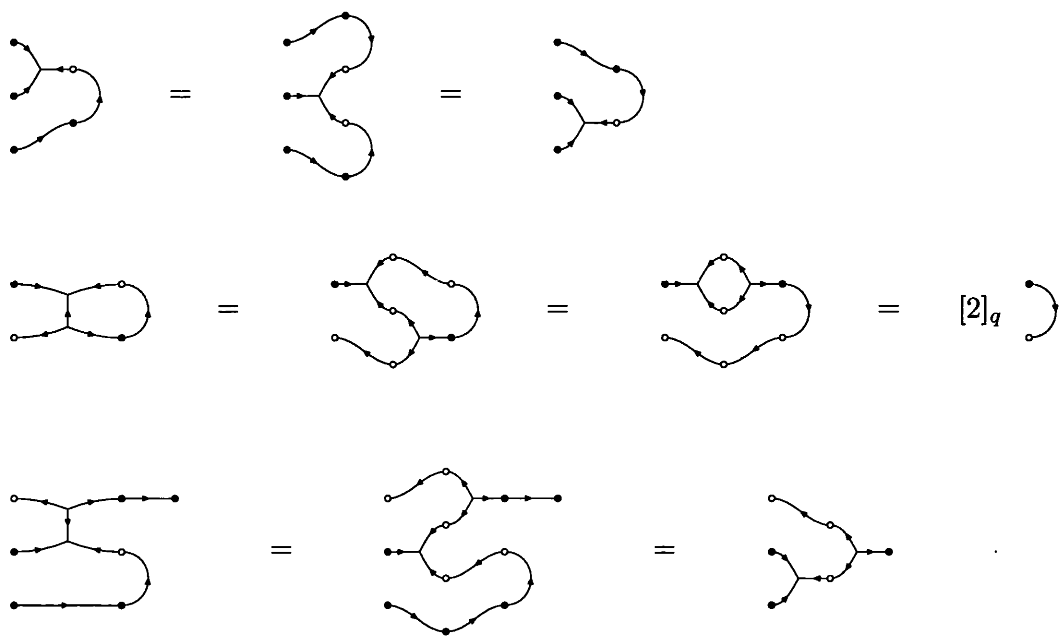
Fix $q \in \mathbb{C} \setminus \{0\}$. Morphisms in $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ are equivalence classes of $\mathbb{C}$-linear combinations of diagrams modulo the tensor ideal generated by the following basic relations and closed under the operations of conjugation, duality and opposite:



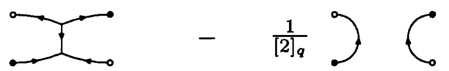
For example, applying *opposite* and *conjugation* to the third of the above relations yields



There are a number of further relations that will appear frequently:



A particular consequence of the second of these relations is that the twist operators are isomorphisms provided $[2]_q$ is invertible, with inverse (up to colour reversal)



Remark 7.1.1. *The diagram category* $\underline{Fus}_{\mathfrak{sl}_3}$ *appears to be coincide with the category that can be built using Kuperberg's* A_2 *spider ([Kup96]), but I have no rigorous proof of this.*

Remark 7.1.2. *I believe the diagrams are purely combinatoric in nature as in the Temperley-Lieb case. In particular, once the vertices have been fixed on the boundary of the rectangle, I believe there is (up to isotopy) only one planar embedding of the graph in the rectangle. I have no rigorous proof that this is indeed the case.*

There is a notion of parity for objects of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. Let ζ_3 be a primitive cube root of unity and define

$$\begin{aligned} \eta(\underline{0}) &= 1\,, \\ \eta(\underline{1}) &= \zeta_3\,. \end{aligned}$$

Extend η to a function $\mathrm{Obj}(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}) \to \{1, \zeta_3, \zeta_3^2\}$ by the relations

$$\begin{aligned} \eta(A \otimes B) &= \eta(A)\eta(B) , \\ \eta(A^*) &= \eta(A)^{-1} . \end{aligned}$$

Proposition 7.1.3. *Morphisms in $\underline{Fus}_{\mathfrak{sl}_3}$ preserve η.*

Proof. The generating maps in Figure 7-2 preserve η. $\square$

Hence, if $\eta(A) \neq \eta(B)$ then

$$\mathrm{Hom}(A, B) = 0 = \mathrm{Hom}(B, A) .$$

There is a $\mathbb{Z}_{\geq 0}$-grading on objects, called the DEGREE, defined by

$$\begin{aligned} \deg\colon \mathrm{Obj}(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}) &\longrightarrow \mathbb{Z}_{\geq 0} \\ \underline{0} &\longmapsto 0 \\ \underline{1} &\longmapsto 1 \\ \underline{1}^* &\longmapsto 1 \end{aligned}$$

and

$$\deg(A \otimes B) = \deg(A) + \deg(B) .$$

This induces a filtration on $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ defined by

$$\mathcal{F}_n = \{\text{morphisms } f \ : \ f \text{ factors through an object of degree } n\} .$$

Let f be a diagram from A to B and consider the set of expressions for f as a composite of the tensor extensions of the list of generators in Fig 7-2. Each such expression $f = \alpha_1 \circ \cdots \circ \alpha_k$ defines a sequence x_i, where each x_i is the degree of the codomain of α_i. The diagram f is said to be CONCAVE if there is an index n such that

$$\deg(A) \geq x_1 \geq \ldots \geq x_{n-1} \geq x_n \leq x_{n+1} \leq \ldots \leq x_{k-1} \leq x_k = \deg(B) .$$

The following result is crucial to the inductive analysis of the category pursued in the next section.

Proposition 7.1.4. *Every morphism ϕ can be expressed as a $\mathbb{C}$-linear combination of concave diagrams.*

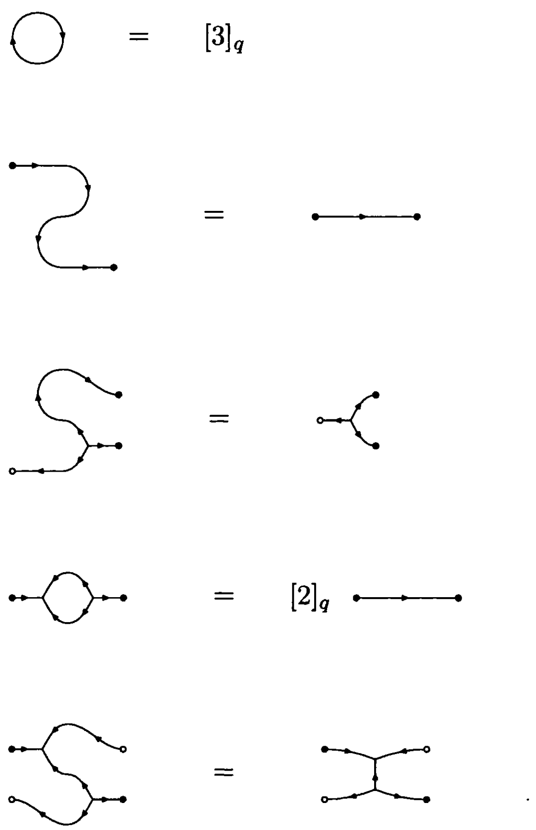
Proof. It is sufficient to prove the result for diagrams. Let f be a diagram from A to B and choose an expression for f as a composite of tensor extensions of the list of generators in Figure 7-2:

$$f = \alpha_1 \circ \cdots \circ \alpha_k \, .$$

Notice that it is sufficient to prove the result for the case $\deg(A) < x_1 = x_2 = \cdots = x_{k-2} = x_{k-1} > x_k = \deg(B)$ with $\deg(A), \deg(B) \in \{x_1 - 2, x_1 - 1\}$. This is because any non-concave diagram contains at least one such sequence of generators and can be made concave by inductively replacing such a sequence by an equivalent concave expression. The algorithm terminates since the sum of the x_i after each iteration is finite, strictly decreasing and bounded below.

Suppose therefore, that $f = \alpha_1 \circ \cdots \circ \alpha_k$ satisfies $\deg(A) < x_1 = \cdots = x_{k-1} > x_k = \deg(B)$ and $\deg(A), \deg(B) \in \{x_1 - 2, , x_1 - 1\}$. The proof proceeds by induction on k, the length of the sequence of generators. Notice that the generators $\alpha_2, \ldots, \alpha_{k-1}$ must be (tensor extensions of) twist operators since these are the only generators for which the domain and codomain have the same degree. The generator α_1 must be (a tensor extension of) a creation operator and the generator α_k must be (a tensor extension of) an annihilation operator.

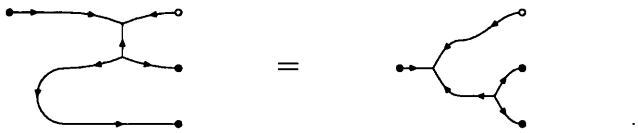
Let $k = 2$. If the creation and annihilation operators are parallel then the order can be reversed, whence the result. Otherwise, there are the following possibilities for composing creation and annihilation operators (up to reflections and colour reversal):



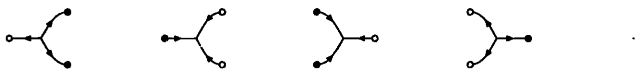
In each case there is an equivalent concave diagram, as required.

Suppose that the result holds for all sequences of the given form with length no greater than $n - 1$ and consider a sequence of length $n \geq 3$. If the creation and annihilation operator are not in the same connected component then the two components can be commuted past each other to yield a concave diagram. If the creation and annihilation operator are in the same connected component, but one of the twist operators is not then this second connected component can be made to operate before (or after) the component containing the creation operator and hence the problem is reduced to that of a sequence of length strictly smaller than n.

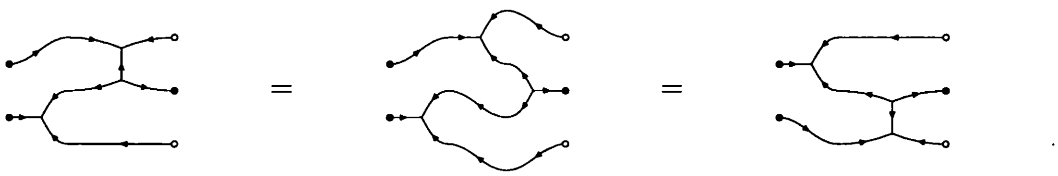
It remains therefore to consider the case when all the generators lie in the same connected component. Suppse that the creation operator is a copairing, which must subsequently be followed by a twist. Then (up to reflection and colour reversal), this composition yields either a multiple of the other copairing or yields



In either case, the resulting diagram has a shorter subsequence of the appropriate form. An identical argument holds if the annihilation operator is a pairing, thus it remains to consider the case when the creation and annihilation operators are the following:

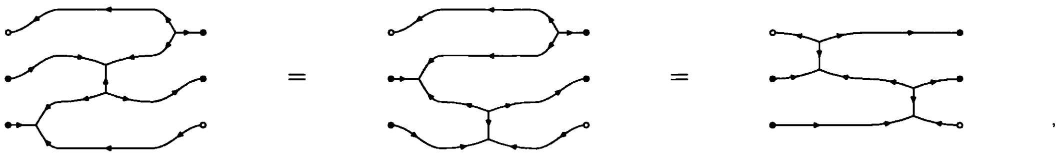


Suppose, without loss of generality, that the creation operator is . Then this must be followed by a twist operator since $n \geq 3$. However, both composites are equal as the following calculation demonstrates:



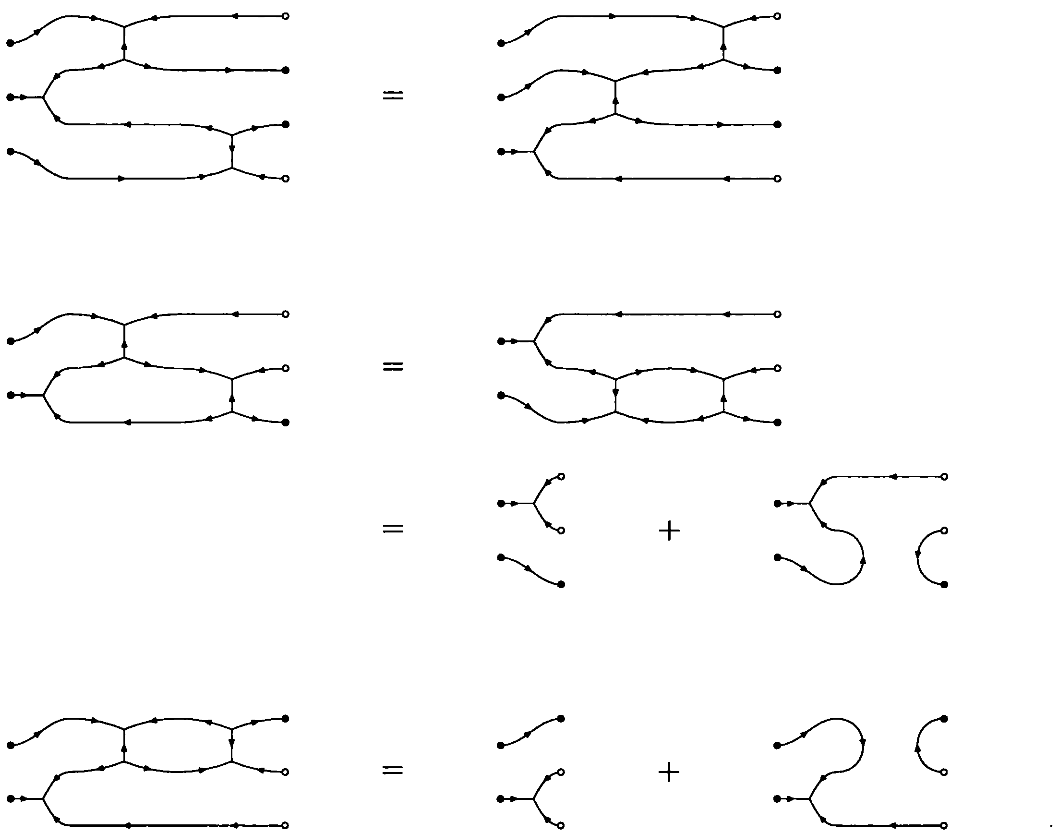
(7.1)

If $n = 3$ then there must now follow an annihilation operator. However the only way of placing an annihilation operator in the same connected component is the following (up to reflection):



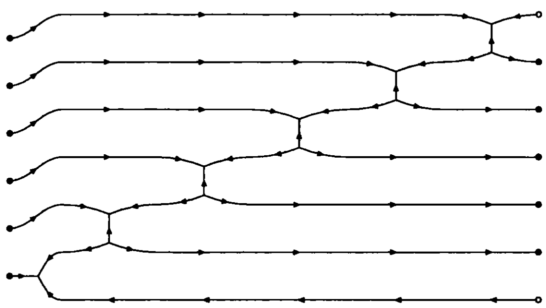
which is now concave.

Suppose now that $n \geq 4$. Then the third operator in the sequence is a twist operator and there are four admissible ways of composing the first three operators:

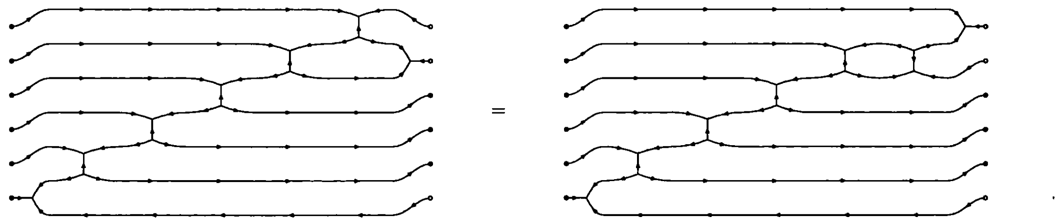


The first two ways are equal as the first line shows. The remaining two possibilities are equal to a linear combination of diagrams that can now be made concave by the induction hypothesis.

It is a simple induction using equation (7.1) and the induction hypothesis to show that it only remains to consider the case where the creation and twist operators compose in the following way:



Now consider the possibilities for where the annihilation operator can be composed. If the annihilation operator operates on the bottom white strand then it commutes past all the twist operators and the result holds by the induction hypothesis. If the annihilation operator operates on the top white strand then moving the twist operators to below the creation operator using (7.1) yields an analogous scenario. If the annihilation operator operates on two of the black strands between the white dots, but not on the top black strand, then it can be commuted past (at least) the last twist operator and the result holds by appealing to the induction hypothesis. Thus the only remaining possibility is that the annihilation operator operates on the top two black strands between the white dots. But now



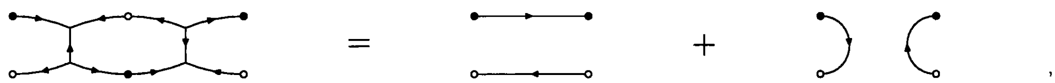
which can be expressed as a linear combination of diagrams that can be made concave by the induction hypothesis. □

The section is concluded by another result that will be crucial to the analysis of the diagram category - this result will be used to prove that the indecomposables defined inductively are bricks in the functor category. For an object A, define $\mathcal{F}_k(A) = \mathcal{F}_k \cap \mathrm{End}(A)$, the endomorphisms of A factoring through an object of degree k.

Lemma 7.1.5. *Let A be an object of $\underline{Fus}_{\mathfrak{sl}_3}$ with $\deg(A) = n$. Then,*

$$\frac{\mathcal{F}_n(A)}{\mathcal{F}_{n-1}(A)} \cong \mathbb{C}\,.$$

Proof. It is sufficient to take γ to be a concave diagram. Choose a concave expression for γ as a composite of generators and identity maps. If an annihilation operator appears in the expression then $\gamma \in \mathcal{F}_{n-1}(A)$ since the codomain of the annihilation operator has degree at most $n-1$. Suppose therefore that the expression for $\gamma \neq \mathrm{id}_A$ contains only twist maps. Each twist map induces a permutation on the string of black and white dots, operating by elementary transposition. Since $\gamma \in \mathrm{End}(A)$, the permutation induced by γ must be the identity. In particular, commuting parallel twists past each other if necessary, the expression for γ must contain a tensor extension of the operator



whose image in $\frac{\mathcal{F}_n(A)}{\mathcal{F}_{n-1}(A)}$ is equal to the image of the identity map. $\square$

7.2 Analysis of the diagram category

7.2.1 Analysis of the category ′

Let $q \in \mathbb{C}\backslash\{0\}$ and consider the diagram category $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. Using the Yoneda functor, embed $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ in the functor category $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$ consisting of $\mathbb{C}$-linear contravariant functors from the diagram category $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ to the category of $\mathbb{C}$-vector spaces. Henceforth, the parameter q will be suppressed in the notation for the quantum integers, so $[3]_q$ will be denoted $[3]$.

The monoidal structure on $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ induces a monoidal structure on $\yen(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3})$, which can also be extended to the image of idempotents as in Section 5.4.2.

It is shown that for generic q, every object in $\yen(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3})$ is semisimple in $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$. The analysis begins by identifying a distinguished collection of indecomposables in $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$.

Theorem 7.2.1. *Suppose $[k] \neq 0$ for every $1 \leq k \leq p+1$. Then there is a collection of bricks $\{X_{m,n} \ : \ m,n \in \mathbb{N} \cup \{0\}$ and $m+n \leq p\}$ in $\mathfrak{Fun}^\circ(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$ satisfying the fusion rule*

$$X_{m,n} \otimes X_{0,1} \cong X_{m-1,n} \oplus X_{m+1,n-1} \oplus X_{m,n+1} \ , \quad 0 \leq m+n \leq p-1 \ , \qquad (7.2)$$

with the convention that $X_{i,j} = 0$ in equation (7.2) if $i < 0$ or $j < 0$.

Moreover, the $X_{m,n}$ form a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$ generated by direct summands of $\yen(\underline{m} \otimes \underline{n}^*)$ with $m+n \leq p$.

Proof. The proof proceeds by induction on the degree $p = m+n$. There are a number of initial cases to consider before the induction step can be proved.

Define $X_{0,0} = \mathrm{Im}(\mathrm{id}_{\underline{0}}) = \yen(\underline{0})$, $X_{1,0} = \mathrm{Im}(\mathrm{id}_{\underline{1}}) = \yen(\underline{1})$ and $X_{0,1} = \mathrm{Im}(\mathrm{id}_{\underline{1}^*}) = \yen(\underline{1}^*)$. These are certainly bricks and have no non-zero morphisms between them by parity. Moreover,

$$X_{0,0} \otimes X_{1,0} = \mathrm{Im}(\mathrm{id}_{\underline{0}} \otimes \mathrm{id}_{\underline{1}}) = \mathrm{Im}(\mathrm{id}_{\underline{1}}) = X_{1,0}$$

and certainly $X_{0,0}$, $X_{1,0}$ and $X_{0,1}$ form a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$ generated by (direct summands of) $\yen(\underline{0})$, $\yen(\underline{1})$

and $¥(\underline{1}^*)$.

The next step is to construct bricks $X_{2,0}$, $X_{1,1}$ and $X_{0,2}$ in degree 2.

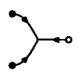
Consider $\mathrm{End}(X_{1,0} \otimes X_{1,0}) = \mathrm{End}(¥(\underline{2})) \cong \mathrm{End}(\underline{2})$. Recall that $\mathrm{End}(\underline{2})$ has a filtration

$$0 = \mathcal{F}_0(\underline{2}) \subset \mathcal{F}_1(\underline{2}) \subset \mathcal{F}_2(\underline{2}) = \mathrm{End}(\underline{2}) \ .$$

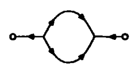
In particular, composition of morphisms defines a surjection

$$\bigoplus_{m+n=k} \mathrm{Hom}(\underline{2}, \underline{m} \otimes \underline{n}^*) \otimes \mathrm{Hom}(\underline{m} \otimes \underline{n}^*, \underline{2}) \longrightarrow \mathcal{F}_k(\underline{2}) \ .$$

Consider therefore $\mathrm{Hom}(\underline{2}, \underline{1}^*)$. As a consequence of Proposition 7.1.4, this has a $\mathbb{C}$-basis given by the diagram

ϕ =  .

Likewise $\mathrm{Hom}(\underline{1}^*, \underline{2})$ has a $\mathbb{C}$-basis given by ϕ°. Notice that $\mathrm{Hom}(\underline{2}, \underline{m} \otimes \underline{n}^*)$ contains only the zero map whenever $(m, n) \neq (0, 1)$ and $m + n \leq 1$. In particular, $\mathcal{F}_1(\underline{2})$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$. Moreover,

$\phi^\circ\phi$ =  = $[2]$,

whence, provided $[2] \neq 0$, the morphism $\frac{1}{[2]}\phi\phi^\circ$ is an idempotent in $\mathrm{End}(\underline{2})$. Define $e_{2,0}$ to be the complementary idempotent in $\mathrm{End}(\underline{2})$:

$$e_{2,0} = \mathrm{id}_{\underline{2}} - \frac{1}{[2]}\phi\phi^\circ \ .$$

Then $e_{2,0}$ satisfies

$$e_{2,0} \,\mathrm{Hom}(\underline{2}, \underline{m} \otimes \underline{n}^*) = 0 = \mathrm{Hom}(\underline{m} \otimes \underline{n}^*, \underline{2}) e_{2,0} \ , \quad \forall m + n \leq 1 \ , \tag{7.3}$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_2(\underline{2})}{\mathcal{F}_1(\underline{2})} \cong \mathbb{C}$. Define $X_{2,0} = \mathrm{Im}((e_{2,0})_*)$. Then, by (7.3),

$$\mathrm{End}(X_{2,0}) \cong e_{2,0} \,\mathrm{End}(\underline{2}) e_{2,0} = \mathbb{C}e_{2,0} \ ,$$

whence $X_{2,0}$ is a brick. By construction

$$X_{1,0} \otimes X_{1,0} \cong X_{0,1} \oplus X_{2,0} \ ,$$

with

$$\begin{aligned}
\mathrm{Hom}(X_{0,1}, X_{1,0} \otimes X_{1,0}) &= \mathbb{C}(\phi^\circ)_* \,, \\
\mathrm{Hom}(X_{2,0}, X_{1,0} \otimes X_{1,0}) &= \mathbb{C}(e_{2,0})_* \,.
\end{aligned}$$

Furthermore, it is clear that there are no non-zero morphisms between $X_{2,0}$ and any of the previously defined bricks $X_{0,0}$, $X_{1,0}$ or $X_{0,1}$. Explicitly, let $\gamma \in \mathrm{Hom}(X_{2,0}, X_{m,n})$ with $(m,n) \in \{(0,0),(1,0),(0,1)\}$. Then $\gamma = g_*$ for some $g = e_{2,0}g$. But g must factor through an object $\underline{m} \otimes \underline{n}^*$ of degree $m+n \leq 1$. Relation (7.3) now asserts g is therefore the zero morphism. The argument in the opposite direction is entirely analogous.

Finally, $\yen(\underline{2}) \cong X_{0,1} \oplus X_{2,0}$, whence $X_{0,0}$, $X_{1,0}$, $X_{0,1}$ and $X_{2,0}$ is a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}_{\mathfrak{sl}_3}}, \underline{\mathrm{Vect}})$ generated by direct summands of $\yen(\underline{0})$, $\yen(\underline{1})$, $\yen(\underline{1}^*)$ and $\yen(\underline{2})$.

Consider next $\mathrm{End}(X_{1,0} \otimes X_{0,1}) \cong \mathrm{End}(\underline{1} \otimes \underline{1}^*)$. Now, $\mathrm{End}(\underline{1} \otimes \underline{1}^*)$ has a filtration

$$\mathcal{F}_0(\underline{1} \otimes \underline{1}^*) \subset \mathcal{F}_1(\underline{1} \otimes \underline{1}^*) \subset \mathcal{F}_2(\underline{1} \otimes \underline{1}^*) = \mathrm{End}(\underline{1} \otimes \underline{1}^*)$$

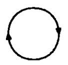
and composition of morphisms defines a surjection

$$\bigoplus_{m+n=k} \mathrm{Hom}(\underline{1} \otimes \underline{1}^*, \underline{m} \otimes \underline{n}^*) \otimes \mathrm{Hom}(\underline{m} \otimes \underline{n}^*, \underline{1} \otimes \underline{1}^*) \longrightarrow \mathcal{F}_k(\underline{1} \otimes \underline{1}^*) \,.$$

Consider therefore $\mathrm{Hom}(\underline{1} \otimes \underline{1}^*, \underline{0})$. By Proposition 7.1.4, this must have a $\mathbb{C}$-basis given by the diagram

ϕ =  .

Likewise $\mathrm{Hom}(\underline{0}, \underline{1} \otimes \underline{1}^*)$ has a $\mathbb{C}$-basis given by ϕ°. Notice that $\mathrm{Hom}(\underline{1} \otimes \underline{1}^*, \underline{m} \otimes \underline{n}^*)$ contains only the zero map whenever $(m,n) \neq (0,0)$ and $m+n \leq 1$. In particular, $\mathcal{F}_0(\underline{1} \otimes \underline{1}^*)$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$. Moreover,

$\phi^\circ\phi$ =  = $[3]$,

whence, provided $[3] \neq 0$, the morphism $\frac{1}{[3]}\phi\phi^\circ$ is an idempotent in $\mathrm{End}(\underline{1} \otimes \underline{1}^*)$. Define $e_{1,1}$ to be the complementary idempotent in $\mathrm{End}(\underline{1} \otimes \underline{1}^*)$:

$$e_{1,1} = \mathrm{id}_{\underline{1} \otimes \underline{1}^*} - \frac{1}{[3]}\phi\phi^\circ \,.$$

Then $e_{1,1}$ satisfies

$$e_{1,1}\,\mathrm{Hom}(\underline{1}\otimes\underline{1}^*,\underline{m}\otimes\underline{n}^*)=0=\mathrm{Hom}(\underline{m}\otimes\underline{n}^*,\underline{1}\otimes\underline{1}^*)e_{1,1}\ ,\quad \forall m+n\leq 1\ , \tag{7.4}$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_2(\underline{1}\otimes\underline{1}^*)}{\mathcal{F}_1(\underline{1}\otimes\underline{1}^*)}\cong\mathbb{C}$. Define $X_{1,1}=\mathrm{Im}((e_{1,1})_*)$. Then, by (7.4),

$$\mathrm{End}(X_{1,1})\cong e_{1,1}\,\mathrm{End}(\underline{1}\otimes\underline{1}^*)e_{1,1}=\mathbb{C}e_{1,1}\ ,$$

whence $X_{1,1}$ is a brick. By construction,

$$X_{1,0}\otimes X_{0,1}\cong X_{0,0}\oplus X_{1,1}\ ,$$

with

$$\begin{aligned}\mathrm{Hom}(X_{0,0},X_{1,0}\otimes X_{0,1}) &= \mathbb{C}(\phi^\circ)_*\ ,\\ \mathrm{Hom}(X_{1,1},X_{1,0}\otimes X_{0,1}) &= \mathbb{C}(e_{1,1})_*\ .\end{aligned}$$

Furthermore, it is clear that there are no non-zero morphisms between $X_{1,1}$ and the other bricks previously defined. The argument for bricks $X_{m,n}$ with $m+n\leq 1$ is identical to the argument explicited in the construction of $X_{2,0}$. It remains to consider $\mathrm{Hom}(X_{1,1},X_{2,0})$. It is clearly possible to appeal to parity to argue that this must contain only the zero map, however this will not be sufficient for larger values of m and n considered later. A better argument is that the any operator capable of *changing* the number of black and white dots must factor through an object of degree strictly less than 2. Relation (7.4) can now be applied yielding the desired result.

Finally, $\yen(\underline{1}\otimes\underline{1}^*)\cong X_{0,0}\oplus X_{1,1}$, whence $X_{0,0}$, $X_{1,0}$, $X_{0,1}$, $X_{2,0}$ and $X_{1,1}$ is a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3},\underline{\mathrm{Vect}})$ generated by direct summands of $\yen(\underline{0})$, $\yen(\underline{1})$, $\yen(\underline{1}^*)$, $\yen(\underline{2})$ and $\yen(\underline{1}\otimes\underline{1}^*)$.

To conclude the constructions in degree 2, consider $\mathrm{End}(X_{0,1}\otimes X_{0,1})\cong\mathrm{End}(\underline{2}^*)$. Notice that the calculations for $X_{2,0}$ hold with colours reversed, thus it is possible to construct a brick $X_{0,2}=\mathrm{Im}((e_{0,2})_*)$, where $e_{0,2}$ is the colour-reversed analogue of $e_{2,0}$, that is, $e_{0,2}=\overline{e_{2,0}}$. Furthermore,

$$X_{0,1}\otimes X_{0,1}\cong X_{1,0}\oplus X_{0,2}$$

and the bricks $\{X_{m,n}:m+n\leq 2\}$ form a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3},\underline{\mathrm{Vect}})$ generated by direct summands of $\{\yen(\underline{m}\otimes\underline{n}^*)\ :\ m+n\leq 2\}$.

There is one final case to consider before moving on to the inductive steps. Consider $\mathrm{End}(X_{1,1}\otimes X_{0,1}) \cong (e_{1,1}\otimes 1)\,\mathrm{End}(\underline{1}\otimes\underline{2}^*)(e_{1,1}\otimes 1)$. The filtration on $\mathrm{End}(\underline{1}\otimes\underline{2}^*)$ induces a filtration

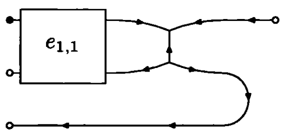
$$0 = (e_{1,1}\otimes 1)\mathcal{F}_0(\underline{1}\otimes\underline{2}^*)(e_{1,1}\otimes 1) \subset \cdots \subset (e_{1,1}\otimes 1)\,\mathrm{End}(\underline{1}\otimes\underline{2}^*)(e_{1,1}\otimes 1)\,.$$

Consider first $(e_{1,1}\otimes 1)\mathcal{F}_1(\underline{1}\otimes\underline{2}^*)(e_{1,1}\otimes 1)$. Composition of morphisms defines a surjection

$$\bigoplus_{m+n\leq 1} (e_{1,1}\otimes 1)\,\mathrm{Hom}(\underline{1}\otimes\underline{2}^*,\underline{m}\otimes\underline{n}^*)\otimes\mathrm{Hom}(\underline{m}\otimes\underline{n}^*,\underline{1}\otimes\underline{2}^*)(e_{1,1}\otimes 1) \longrightarrow \mathcal{F}_1(\underline{1}\otimes\underline{2}^*)\,,$$

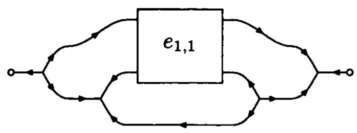
so consider $(e_{1,1}\otimes 1)\,\mathrm{Hom}(\underline{1}\otimes\underline{2}^*,\underline{1}^*)$. As a consequence of Proposition 7.1.4, this is no more than one-dimensional, since the only composition of $e_{1,1}\otimes 1$ with a morphism in $\mathrm{Hom}(\underline{1}\otimes\underline{2}^*,\underline{1}^*)$ that is not obviously zero is the following:

ϕ =



Strictly speaking, it is not clear that ϕ is non-zero, but this will be shown shortly. Likewise $\mathrm{Hom}(\underline{1}^*,\underline{1}\otimes\underline{2}^*)(e_{1,1}\otimes 1) = \mathbb{C}\phi^\circ$. Notice that $\mathrm{Hom}(\underline{1}\otimes\underline{2}^*,\underline{m}\otimes\underline{n}^*)$ contains only the zero map whenever $(m,n)\neq(0,1)$ and $m+n\leq 1$. In particular, $(e_{1,1}\otimes 1)\mathcal{F}_1(\underline{1}\otimes\underline{2}^*)(e_{1,1}\otimes 1)$ is spanned by $\phi\phi^\circ$. Consider the following calculation:

$\phi^\circ\phi$ =



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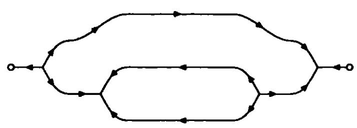
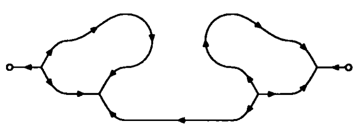


$-\frac{1}{[3]}$



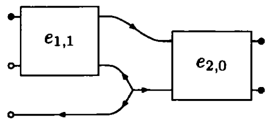
$= \left([2]^2 - \frac{[2]^2}{[3]}\right)$ $= \frac{[2][4]}{[3]}$.

Thus, provided $[4] \neq 0$, the operator $\frac{[3]}{[2][4]}\phi\phi^\circ$ is an idempotent in $(e_{1,1} \otimes 1)\mathcal{F}_1(\underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1)$. Notice that this also establishes that $\phi\phi^\circ \neq 0$ provided $[4] \neq 0$, since multiplying on the left by ϕ° and on the right by ϕ yields a non-zero multiple of $\mathrm{id}_{\underline{1}^*}$. Thus, $(e_{1,1} \otimes 1)\mathcal{F}_1(\underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1)$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$.

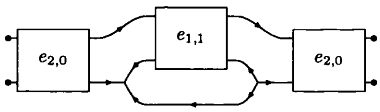
Consider next $(e_{1,1} \otimes 1)\mathcal{F}_2(\underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1)$. Then there is a surjection

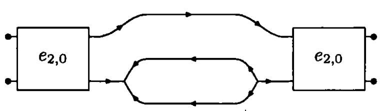
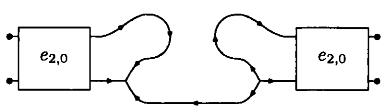
$$\bigoplus_{m+n\leq 2} (e_{1,1} \otimes 1)\,\mathrm{Hom}(\underline{1} \otimes \underline{2}^*, \underline{m} \otimes \underline{n}^*) \otimes \mathrm{Hom}(\underline{m} \otimes \underline{n}^*, \underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1) \longrightarrow \mathcal{F}_2(\underline{1} \otimes \underline{2}^*)\,.$$

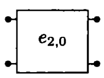
Since $\mathcal{F}_1(\underline{1} \otimes \underline{2}^*) \subset \mathcal{F}_2(\underline{1} \otimes \underline{2}^*)$ has already been considered, consider $(e_{1,1} \otimes 1)\,\mathrm{Hom}(\underline{1} \otimes \underline{2}^*, \underline{2})e_{2,0}$. As a consequence of Proposition 7.1.4, this is no more than one-dimensional, since the only composition of $e_{1,1} \otimes 1$ with a morphism in $\mathrm{Hom}(\underline{1} \otimes \underline{2}^*, \underline{2})e_{2,0}$ that is not obviously zero is the following:

ψ =  .

Strictly speaking, it is not clear that ψ is non-zero, but this will be shown shortly. Likewise $e_{2,0}\,\mathrm{Hom}(\underline{2}, \underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1) = \mathbb{C}\psi^\circ$. Notice that $\mathrm{Hom}(\underline{1} \otimes \underline{2}^*, \underline{m} \otimes \underline{n}^*)$ contains only the zero map whenever $(m, n) \neq (2, 0)$ and $m + n = 2$. In particular, $(e_{1,1} \otimes 1)\mathcal{F}_2(\underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1)$ is spanned by $\psi\psi^\circ$ and $\phi\phi^\circ$. Consider the following calculation:

$\psi^\circ\psi$ = 

=  $-\frac{1}{[3]}$ 

= $[2]$  .

Hence $\frac{1}{[2]}\psi\psi^\circ$ is an idempotent in $(e_{1,1} \otimes 1)\mathcal{F}_2(\underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1)$. Notice that this also establishes that $\psi\psi^\circ \neq 0$, since multiplying on the left by ψ° and on the right by ψ yields a non-zero multiple of $e_{2,0} \neq 0$. Thus, $(e_{1,1} \otimes 1)\mathcal{F}_2(\underline{1} \otimes \underline{2}^*)(e_{1,1} \otimes 1)$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$ and $\psi\psi^\circ$. Define

$$e_{1,2} = e_{1,1} \otimes 1 - \frac{1}{[2]}\psi\psi^\circ - \frac{[3]}{[2][4]}\phi\phi^\circ\,.$$

Then the idempotent $e_{1,2}$ satisfies

$$e_{1,2}\,\mathrm{Hom}(\underline{1}\otimes\underline{2}^*,\underline{m}\otimes\underline{n}^*) = 0 = \mathrm{Hom}(\underline{m}\otimes\underline{n}^*,\underline{1}\otimes\underline{2}^*)e_{1,2}\ , \quad \forall\, m+n\leq 2\ , \qquad (7.5)$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_3(\underline{1}\otimes\underline{2}^*)}{\mathcal{F}_2(\underline{1}\otimes\underline{2}^*)} \cong \mathbb{C}$. Define $X_{1,2} = \mathrm{Im}((e_{1,2})_*)$. Then, by (7.5),

$$\mathrm{End}(X_{1,2}) \cong e_{1,2}\,\mathrm{End}(\underline{1}\otimes\underline{2}^*)e_{1,2} = \mathbb{C}e_{1,2}\ ,$$

whence $X_{1,2}$ is a brick. By construction,

$$X_{1,1}\otimes X_{0,1} \cong X_{0,1}\oplus X_{2,0}\oplus X_{1,2}\ ,$$

with

$$\begin{aligned}
\mathrm{Hom}(X_{0,1},X_{1,1}\otimes X_{0,1}) &= \mathbb{C}(\phi^\circ)_*\ ,\\
\mathrm{Hom}(X_{2,0},X_{1,1}\otimes X_{0,1}) &= \mathbb{C}(\psi^\circ)_*\ ,\\
\mathrm{Hom}(X_{1,2},X_{1,1}\otimes X_{0,1}) &= \mathbb{C}(e_{1,2})_*\ .
\end{aligned}$$

Furthermore, it is clear that there are no non-zero morphisms between $X_{1,2}$ and the other bricks previously defined.

Finally, $\yen(\underline{1}\otimes\underline{2}^*) = \yen(\underline{1}\otimes\underline{1}^*)\otimes X_{0,1} \cong X_{0,0}\otimes X_{0,1}\oplus X_{1,1}\otimes X_{0,1}$, whence $X_{0,0}$, $X_{1,0}$, $X_{0,1}$, $X_{2,0}$, $X_{1,1}$, $X_{0,2}$ and $X_{1,2}$ is a complete set of indecomposables for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3},\underline{\mathrm{Vect}})$ generated by direct summands of $\yen(\underline{0})$, $\yen(\underline{1})$, $\yen(\underline{1}^*)$, $\yen(\underline{2})$, $\yen(\underline{1}\otimes\underline{1}^*)$, $\yen(\underline{2}^*)$ and $\yen(\underline{1}\otimes\underline{2}^*)$.

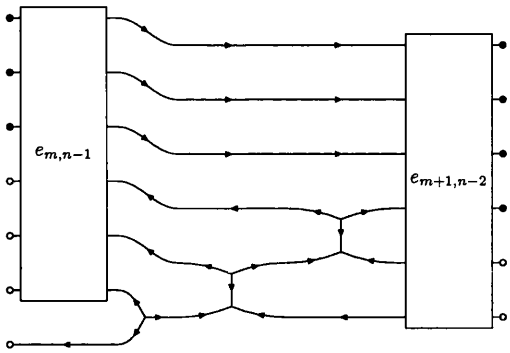
It is now possible to proceed to the inductive steps. There are four variations on the same theme: three dealing with $X_{i,j}$ for small values of i or j, which require special treatment, and one dealing with the case when i and j are both sufficiently large.

Assume that bricks $X_{m,n}$ satisfying the hypotheses of the theorem have been constructed for all pairs $(m,n)\in(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\})$ satisfying $m+n\leq p$ for some fixed p. Suppose further that $X_{m,n} = \mathrm{Im}((e_{m,n})_*)$ for some non-zero idempotent $e_{m,n}$ defined by the formula

$$e_{m,n} \;=\; e_{m,n-1}\otimes\underline{1}^* - \frac{[n-1]}{[n]}\psi\psi^\circ - \frac{[m][m+n]}{[m+1][m+n+1]}\phi\phi^\circ\ ,$$

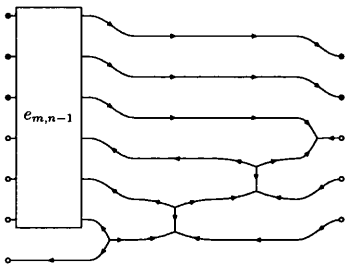
where

ψ =



and

ϕ =



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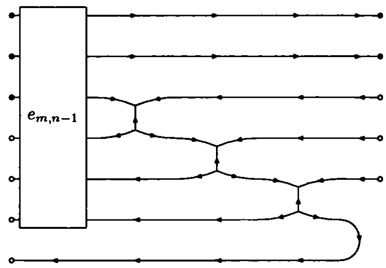


.

To conclude the induction, the constructions in degree $p+1$ are considered.

Let $m \geq 2$ and consider first $\mathrm{End}(X_{m,0} \otimes X_{1,0}) \cong (e_{m,0} \otimes 1)\,\mathrm{End}(\underline{m+1})(e_{m,0} \otimes 1)$. The filtration on $\mathrm{End}(\underline{m+1})$ induces a filtration

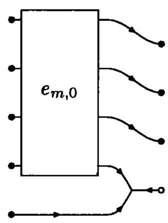
$$0 = (e_{m,0} \otimes 1)\mathcal{F}_0(\underline{m+1})(e_{m,0} \otimes 1) \subset \cdots \subset (e_{m,0} \otimes 1)\,\mathrm{End}(\underline{m+1})(e_{m,0} \otimes 1)\ .$$

Consider $(e_{m,0} \otimes 1)\mathcal{F}_m(\underline{m+1})(e_{m,0} \otimes 1)$. Composition of morphisms defines a surjection

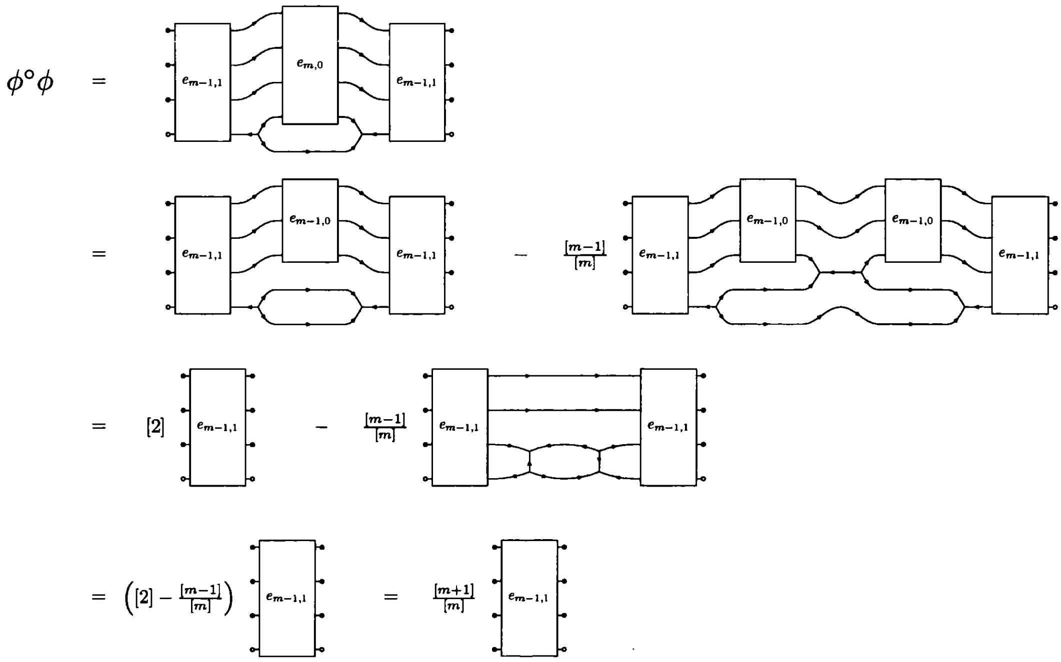
$$\bigoplus_{r+s \leq m} (e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m+1}, \underline{r} \otimes \underline{s}^*) \otimes \mathrm{Hom}(\underline{r} \otimes \underline{s}^*, \underline{m+1})(e_{m,0} \otimes 1) \longrightarrow \mathcal{F}_m(\underline{m+1})\ .$$

Consider therefore $(e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m+1}, \underline{m-1} \otimes \underline{1}^*)$. As a consequence of Proposition 7.1.4, this is no more than one-dimensional, since the only composition of $e_{m,0} \otimes 1$ with a morphism in $\mathrm{Hom}(\underline{m+1}, \underline{m-1} \otimes \underline{1}^*)$ that is not obviously zero is the following:

ϕ =



Strictly speaking, it is not clear that ϕ is non-zero, but this will be shown shortly. Notice that, since $(e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m+1}, \underline{r} \otimes \underline{s}^*) = 0$ whenever $\underline{r} + \underline{s} \leq m-1$, the map ϕ must satisfy $\phi = \phi e_{m-1,1}$. Likewise $\mathrm{Hom}(\underline{m-1} \otimes \underline{1}^*, \underline{m+1})(e_{m,0} \otimes 1) = \mathbb{C}\phi^\circ$. In particular, $(e_{m,0} \otimes 1)\mathcal{F}_m(\underline{m+1})(e_{m,0} \otimes 1)$ is spanned by $\phi\phi^\circ$. Consider the following calculation:



Thus, provided $[m+1] \neq 0$, the operator $\frac{[m]}{[m+1]}\phi\phi^\circ$ is an idempotent in $(e_{m-1,1} \otimes 1)\mathcal{F}_m(\underline{m+1})(e_{m-1,1} \otimes 1)$. This also establishes that $\phi\phi^\circ \neq 0$. Hence, $(e_{m,0} \otimes 1)\mathcal{F}_m(\underline{m+1})(e_{m,0} \otimes 1)$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$. Define

$$e_{m+1,0} = e_{m-1,1} \otimes 1 - \frac{[m]}{[m+1]}\phi\phi^\circ \; .$$

Then the idempotent $e_{m+1,0}$ satisfies

$$e_{m+1,0}\,\mathrm{Hom}(\underline{m+1}, \underline{r} \otimes \underline{s}^*) = 0 = \mathrm{Hom}(\underline{r} \otimes \underline{s}^*, \underline{m+1})e_{m+1,0} \; , \quad \forall\, r+s \leq m \; , \qquad (7.6)$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_{m+1}(\underline{m+1})}{\mathcal{F}_m(\underline{m+1})} \cong \mathbb{C}$. Define $X_{m+1,0} = \mathrm{Im}((e_{m+1,0})_*)$. Then, by (7.6),

$$\mathrm{End}(X_{m+1,0}) \cong e_{m+1,0}\,\mathrm{End}(\underline{m+1})e_{m+1,0} = \mathbb{C}e_{m+1,0} \; ,$$

whence $X_{m+1,0}$ is a brick. By construction,

$$X_{m,0} \otimes X_{1,0} \cong X_{m-1,1} \oplus X_{m+1,0} \ ,$$

with

$$\begin{aligned} \mathrm{Hom}(X_{m-1,1}, X_{m,0} \otimes X_{1,0}) &= \mathbb{C}(\phi^\circ)_* \ , \\ \mathrm{Hom}(X_{m+1,0}, X_{m,0} \otimes X_{1,0}) &= \mathbb{C}(e_{m+1,0})_* \ . \end{aligned}$$

Furthermore, it is clear that

$$\mathrm{Hom}(X_{m+1,0}, X_{r,s}) = 0 = \mathrm{Hom}(X_{r,s}, X_{m+1,0}) \ , \quad \forall r+s \leq m \ .$$

Finally, a simple induction shows that $¥(\underline{m+1}) = ¥(\underline{m}) \otimes X_{1,0}$ has a direct sum decomposition in terms of the $X_{r,s}$ defined thus far.

Notice that a colour-reversed argument defines an idempotent $e_{0,m+1} = \overline{e_{m+1,0}}$ and a brick $X_{0,m+1}$ with the requisite properties. This is perhaps not immediately clear, as there is some concern that in the colour-reversed argument the order of the dots will be white on top of black instead of the other way round. In order to appeal to the induction hypothesis, the black dot appearing in the analysis will have to be "twisted" past the white dots to emerge on the top. However, these twist operators will cancel pairwise in the pertinent calculations, yielding the desired result.

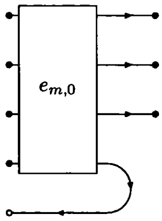
Let $m \geq 2$ and consider next $\mathrm{End}(X_{m,0} \otimes X_{0,1}) \cong (e_{m,0} \otimes 1)\,\mathrm{End}(\underline{m} \otimes \underline{1}^*)(e_{m,0} \otimes 1)$. The filtration on $\mathrm{End}(\underline{m} \otimes \underline{1}^*)$ induces a filtration

$$0 = (e_{m,0} \otimes 1)\mathcal{F}_0(\underline{m} \otimes \underline{1}^*)(e_{m,0} \otimes 1) \subset \cdots \subset (e_{m,0} \otimes 1)\,\mathrm{End}(\underline{m} \otimes \underline{1}^*)(e_{m,0} \otimes 1) \ .$$

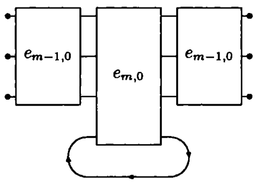
Consider $(e_{m,0} \otimes 1)\mathcal{F}_m(\underline{m} \otimes \underline{1}^*)(e_{m,0} \otimes 1)$. Composition of morphisms defines a surjection

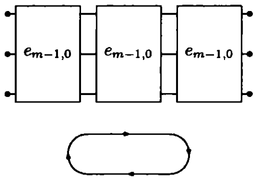
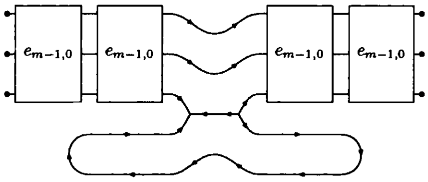
$$\bigoplus_{r+s \leq m} (e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{r} \otimes \underline{s}^*) \otimes \mathrm{Hom}(\underline{r} \otimes \underline{s}^*, \underline{m} \otimes \underline{1}^*)(e_{m,0} \otimes 1) \longrightarrow \mathcal{F}_m(\underline{m} \otimes \underline{1}^*) \ .$$

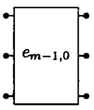
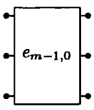
Consider therefore $(e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{m-1})$. As a consequence of Proposition 7.1.4, this is no more than one-dimensional, since the only composition of $e_{m,0} \otimes 1$ with a morphism in $\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{m-1})$ that is not obviously zero is the following:

$$\phi \quad = $$ 

Strictly speaking, it is not clear that ϕ is non-zero, but this will be shown shortly. Notice that, since $(e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{r} \otimes \underline{s}) = 0$ whenever $r + s \leq m - 2$, the map ϕ must satisfy $\phi = \phi e_{m-1,0}$. Likewise $\mathrm{Hom}(\underline{m-1}, \underline{m} \otimes \underline{1}^*)(e_{m,0} \otimes 1) = \mathbb{C}\phi^\circ$. Moreover, $(e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{r} \otimes \underline{s}^*) = 0$ whenever $(r, s) \neq (m-1, 0)$ and $r + s \leq m$. In particular, $(e_{m,0} \otimes 1)\mathcal{F}_m(\underline{m} \otimes \underline{1}^*)(e_{m,0} \otimes 1)$ is spanned by $\phi\phi^\circ$. Consider the following calculation:

$$\phi^\circ\phi \quad = $$ 

$$=$$  $$- \frac{[m-1]}{[m]}$$ 

$$= \left([3] - \frac{[2][m-1]}{[m]}\right)$$  $$= \frac{[m+2]}{[m]}$$ 

Thus, provided $[m+2] \neq 0$, the operator $\frac{[m]}{[m+2]}\phi\phi^\circ$ is an idempotent in $(e_{m,0}\otimes 1)\mathcal{F}_m(\underline{m}\otimes \underline{1}^*)(e_{m,0}\otimes 1)$. This also establishes that $\phi\phi^\circ \neq 0$. Hence, $(e_{m,0}\otimes 1)\mathcal{F}_m(\underline{m}\otimes\underline{1}^*)(e_{m,0}\otimes 1)$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$. Define

$$e_{m,1} = e_{m,0} \otimes 1 - \frac{[m]}{[m+2]}\phi\phi^\circ \; .$$

Then the idempotent $e_{m,1}$ satisfies

$$e_{m,1}\,\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{r} \otimes \underline{s}^*) = 0 = \mathrm{Hom}(\underline{r} \otimes \underline{s}^*, \underline{m} \otimes \underline{1}^*)e_{m,1} \; , \quad \forall r + s \leq m \; , \tag{7.7}$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_{m+1}(\underline{m}\otimes\underline{1}^*)}{\mathcal{F}_m(\underline{m}\otimes\underline{1}^*)} \cong \mathbb{C}$. Define $X_{m,1} = \mathrm{Im}((e_{m,1})_*)$. Then

$$\mathrm{End}(X_{m,1}) \cong e_{m,1}\,\mathrm{End}(\underline{m}\otimes\underline{1}^*)e_{m,1} = \mathbb{C}e_{m,1}\ ,$$

whence $X_{m,1}$ is a brick. By construction,

$$X_{m,0}\otimes X_{0,1} \cong X_{m-1,0}\oplus X_{m,1}\ ,$$

with

$$\begin{aligned} \mathrm{Hom}(X_{m-1,0}, X_{m,0}\otimes X_{0,1}) &= \mathbb{C}(\phi^\circ)_*\ , \\ \mathrm{Hom}(X_{m,1}, X_{m,0}\otimes X_{0,1}) &= \mathbb{C}(e_{m,1})_*\ . \end{aligned}$$

Furthermore, it is clear that

$$\mathrm{Hom}(X_{m,1}, X_{m+1,0}) = 0 = \mathrm{Hom}(X_{m+1,0}, X_{m,1})$$

and

$$\mathrm{Hom}(X_{m,1}, X_{r,s}) = 0 = \mathrm{Hom}(X_{r,s}, X_{m,1})\ , \quad \forall\, r+s\leq m\ .$$

Finally, a simple induction shows that $¥(\underline{m}\otimes\underline{1}^*) = ¥(\underline{m})\otimes X_{0,1}$ has a direct sum decomposition in terms of the $X_{r,s}$ defined thus far.

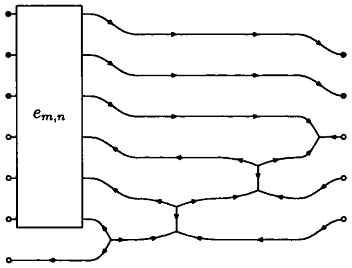
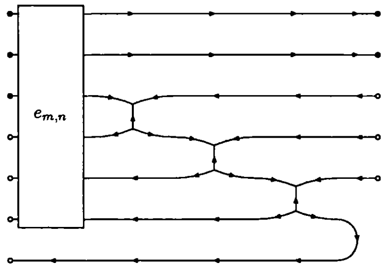
For the final induction, let $m,n\geq 1$ with $(m,n)\neq(1,1)$ and consider $\mathrm{End}(X_{m,n}\otimes X_{0,1}) \cong (e_{m,n}\otimes 1)\,\mathrm{End}(\underline{m}\otimes\underline{n+1}^*)$. The filtration on $\mathrm{End}(\underline{m}\otimes\underline{n+1}^*)$ induces a filtration

$$0 = (e_{m,n}\otimes 1)\mathcal{F}_0(\underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1) \subset \cdots \subset (e_{m,n}\otimes 1)\,\mathrm{End}(\underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1)\ .$$

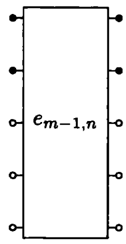
Consider first $(e_{m,n}\otimes 1)\mathcal{F}_{m+n-1}(\underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1)$. Then by Proposition 7.1.4, composition of morphisms defines a surjection

$$\bigoplus_{r+s\leq m+n-1} (e_{m,n}\otimes 1)\,\mathrm{Hom}(\underline{m}\otimes\underline{n+1}^*, \underline{r}\otimes\underline{s}^*)\otimes\mathrm{Hom}(\underline{r}\otimes\underline{s}^*, \underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1)$$
$$\longrightarrow \mathcal{F}_{m+n-1}(\underline{m}\otimes\underline{n+1}^*)\ .$$

Consider therefore $(e_{m,n}\otimes 1)\,\mathrm{Hom}(\underline{m}\otimes\underline{n+1}^*, \underline{m-1}\otimes\underline{n}^*)$. As a consequence of Proposition 7.1.4, this is no more than one-dimensional, since the only composition of $e_{m,n}\otimes 1$ with a morphism in $\mathrm{Hom}(\underline{m}\otimes\underline{n+1}^*, \underline{m-1}\otimes\underline{n}^*)$ that is not obviously zero is the following:

$\phi \quad =$  $=$ 

Strictly speaking, it is not clear that ϕ is non-zero, but this will be shown shortly. Notice that, since $(e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{r} \otimes \underline{s}) = 0$ whenever $r + s \le m - 2$, the map ϕ must satisfy $\phi = \phi e_{m-1,0}$. Likewise $\mathrm{Hom}(\underline{m-1} \otimes \underline{n}^*, \underline{m} \otimes \underline{n+1}^*)(e_{m,n} \otimes 1) = \mathbb{C}\phi^\circ$. Moreover, $(e_{m,0} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{1}^*, \underline{r} \otimes \underline{s}) = 0$ whenever $(r, s) \ne (m-1, n)$ and $r + s \le m - 1$. In particular, $(e_{m,n} \otimes 1)\mathcal{F}_{m+n-1}(\underline{m} \otimes \underline{n+1}^*)(e_{m,n} \otimes 1)$ is spanned by $\phi\phi^\circ$. Now,

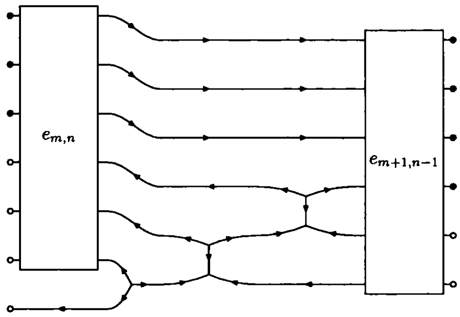
$\phi^\circ\phi \quad = \quad \frac{[m+1][m+n+2]}{[m][m+n+1]}$ 

with the complete calculation detailed in Appendix D Calculation D.1. In particular, provided $[m + n + 2] \ne 0$, the operator $\frac{[m][m+n+1]}{[m+1][m+n+2]}\phi\phi^\circ$ is an idempotent in $(e_{m,n} \otimes 1)\mathcal{F}_{m+n-1}(\underline{m} \otimes \underline{n+1}^*)(e_{m,n} \otimes 1)$. This also establishes that $\phi\phi^\circ \ne 0$. Hence, $(e_{m,n} \otimes 1)\mathcal{F}_{m+n-1}(\underline{m} \otimes \underline{n+1}^*)(e_{m,n} \otimes 1)$ has a $\mathbb{C}$-basis given by $\phi\phi^\circ$.

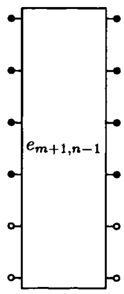
Consider next $(e_{m,n} \otimes 1)\mathcal{F}_{m+n}(\underline{m} \otimes \underline{n+1}^*)(e_{m,n} \otimes 1)$. Composition of morphisms defines a surjection

$$\bigoplus_{r+s \le m+n} (e_{r,s} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{n+1}^*, \underline{r} \otimes \underline{s}^*) \otimes \mathrm{Hom}(\underline{r} \otimes \underline{s}^*, \underline{m} \otimes \underline{n+1}^*)(e_{m,n} \otimes 1) \longrightarrow \mathcal{F}_{m+n}(\underline{m} \otimes \underline{n+1}^*)\ .$$

Since $\mathcal{F}_{m+n-1}(\underline{m} \otimes \underline{n+1}^*) \subset \mathcal{F}_{m+n}(\underline{m} \otimes \underline{n+1}^*)$ has already been considered, consider $(e_{m,n} \otimes 1)\,\mathrm{Hom}(\underline{m} \otimes \underline{n+1}^*, \underline{m+1} \otimes \underline{n-1}^*)e_{m+1,n-1}$. This is no more than one-dimensional, since the only composition of $e_{m,n} \otimes 1$ with a morphism in $\mathrm{Hom}(\underline{m} \otimes \underline{n+1}^*, \underline{m+1} \otimes \underline{n-1}^*)$ that is not obviously zero and is invariant under right composition with $e_{m+1,n-1}$ is the following:

$\psi \quad = \quad$ 

Strictly speaking, it is not clear that ψ is non-zero, but this will be shown shortly. Likewise $e_{m+1,n-2}\,\mathrm{Hom}(\underline{m+1}\otimes\underline{n-1}^*, \underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1) = \mathbb{C}\psi^\circ$. Moreover, $(e_{m,n}\otimes 1)\,\mathrm{Hom}(\underline{m}\otimes\underline{n+1}^*, \underline{r}\otimes\underline{s}^*) = 0$ whenever $(r,s) \neq (m+1, n-1)$ and $r+s = m+n$. In particular, $(e_{m,n}\otimes 1)\mathcal{F}_{m+n}(\underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1)$ is spanned by $\psi\psi^\circ$ and $\phi\phi^\circ$. Now,

$\psi^\circ\psi \quad = \quad \frac{[n+1]}{[n]}$ 

with the complete calculation detailed in Appendix D Calculation D.2. Hence $\frac{[n]}{[n+1]}\psi\psi^\circ$ is an idempotent in $(e_{m,n}\otimes 1)\mathcal{F}_{m+n}(\underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1)$. This also establishes that $\phi\phi^\circ \neq 0$. Hence, $(e_{m,n}\otimes 1)\mathcal{F}_{m+n}(\underline{m}\otimes\underline{n+1}^*)(e_{m,n}\otimes 1)$ has a $\mathbb{C}$-basis given by $\psi\psi^\circ$ and $\phi\phi^\circ$. Define

$$e_{m,n+1} = e_{m,n}\otimes 1 - \frac{[n]}{[n+1]}\psi\psi^\circ - \frac{[m][m+n+1]}{[m+1][m+n+2]}\phi\phi^\circ .$$

Then the idempotent $e_{m,n+1}$ satisfies

$$e_{m,n+1}\,\mathrm{Hom}(\underline{m}\otimes\underline{n+1}^*, \underline{r}\otimes\underline{s}^*) = 0 = \mathrm{Hom}(\underline{r}\otimes\underline{s}^*, \underline{m}\otimes\underline{n+1}^*)e_{m,n+1}\ , \quad \forall\, r+s \leq m+n\ , \tag{7.8}$$

and has a non-zero image in the quotient $\frac{\mathcal{F}_{m+n+1}(\underline{m}\otimes\underline{n+1}^*)}{\mathcal{F}_{m+n}(\underline{m}\otimes\underline{n+1}^*)} \cong \mathbb{C}$. Define $X_{m,n+1} = \mathrm{Im}((e_{m,n+1})_*)$. Then

$$\mathrm{End}(X_{m,n+1}) \cong e_{m,n+1}\,\mathrm{End}(\underline{m}\otimes\underline{n+1}^*)e_{m,n+1} = \mathbb{C}e_{m,n+1}\ ,$$

whence $X_{m,n+1}$ is a brick. By construction,

$$X_{m,n} \otimes X_{0,1} \cong X_{m-1,n} \oplus X_{m+1,n-1} \oplus X_{m,n+1} \ ,$$

with

$$\begin{aligned} \mathrm{Hom}(X_{m-1,n}, X_{m,n} \otimes X_{0,1}) &= \mathbb{C}(\phi^\circ)_* \ , \\ \mathrm{Hom}(X_{m+1,n-1}, X_{m,n} \otimes X_{0,1}) &= \mathbb{C}(\psi^\circ)_* \ , \\ \mathrm{Hom}(X_{m,n+1}, X_{m,n} \otimes X_{0,1}) &= \mathbb{C}(e_{1,2})_* \ . \end{aligned}$$

Furthermore, it is clear that there are no non-zero morphisms between $X_{m,n+1}$ and the other bricks $X_{r,s}$ defined previously.

Finally, a simple induction shows that $¥(\underline{m} \otimes \underline{n+1}^*) = ¥(\underline{m} \otimes \underline{n}^*) \otimes X_{0,1}$ has a direct sum decomposition in terms of the $X_{r,s}$ defined thus far.

This concludes the proof. □

Remark 7.2.2. *For singular q, denote by h the smallest positive integer satisfying $[h] = 0$. The proof of Theorem 7.2.1 permits the construction of bricks $X_{m,n}$ satisfying $m+n \leq h-2$. Notice that it is also possible to construct the bricks $X_{h-1,0}$ and $X_{0,h-1}$.*

The next result shows that $\bullet \otimes X_{1,0}$ behaves in a similar way to $\bullet \otimes X_{0,1}$ of Theorem 7.2.1.

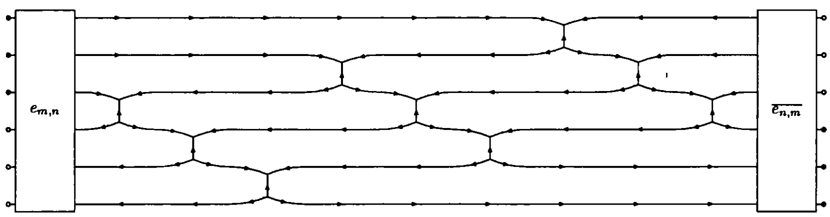
Proposition 7.2.3. *The bricks $X_{m,n}$ $(m, n \geq 0)$ satisfy the fusion rule*

$$X_{m,n} \otimes X_{1,0} \cong X_{m,n-1} \oplus X_{m-1,n+1} \oplus X_{m+1,n} \ ,$$

where $X_{i,j}$ is defined to be zero whenever one of $i < 0$ or $j < 0$.

Proof. Theorem 7.2.1 holds with colours reversed, hence there are idempotents $\overline{e_{n,m}} \in \mathrm{End}(\underline{n}^* \otimes \underline{m})$ such that the image functors $\overline{X}_{n,m} = \mathrm{Im}((\overline{e_{n,m}})_*)$ satisfy the required fusion rule. It remains to relate the $\overline{e_{n,m}}$ to the original $e_{m,n}$. Consider the following composition of morphisms:

ϕ =



Clearly, $\phi_* \in \mathrm{Hom}(X_{m,n}, \overline{X}_{n,m})$. Hence $\phi_*\phi_*^\circ \in \mathrm{End}(X_{m,n})$ so must be a multiple of $(e_{m,n})_*$. The expression of $\overline{e_{n,m}}$ as a linear combination of diagrams contains the identity diagram $\mathrm{id}_{\underline{n}^*\otimes\underline{m}}$ with coefficient 1. Moreover, this is the only diagram not factoring through an object of degree less than $m+n$. Hence $\phi\phi^\circ = e_{m,n}$ and an identical argument yields $\phi^\circ\phi = \overline{e_{n,m}}$. In particular, $X_{m,n} \cong \overline{X}_{n,m}$. □

Much of the analysis performed for the bricks in the Temperley-Lieb chapter also holds for the $X_{m,n}$.

Lemma 7.2.4. *Suppose $[k] \neq 0$ for every $1 \leq k \leq p+1$ and let A be an object of degree p. Suppose η is a non-zero natural transformation $X_{r,s} \to ¥(A)$. Then there exists a natural transformation $\chi\colon ¥(A) \to X_{r,s}$ satisfying $\chi \circ \eta = \mathrm{id}_{X_{r,s}}$. Similarly, there is a right inverse to every non-zero morphism $¥(A) \to X_{r,s}$.*

Proof. Let η be a non-zero transformation $X_{r,s} \to ¥(A)$. The object A is isomorphic to an object of the form $\underline{m} \otimes \underline{n}^*$ for some $m+n=p$, whence A is semisimple. Choose a direct sum decomposition for $¥(A)$ of the form

$$¥(A) = \bigoplus_k X_{m_k,n_k} \; ,$$

including the inclusion maps ι_k and the projection maps π_k. Then $X_{r,s}$ is a direct summand in $¥(A)$ since

$$\eta = \eta\, \mathrm{id}_{¥(A)} = \sum_k \eta\pi_k\iota_k$$

and $\mathrm{Hom}(X_{r,s}, X_{m_k,n_k}) = 0$ whenever $(r,s) \neq (m_k, n_k)$. In particular, there exists some index k for which $\eta\pi_k \neq 0$. However, $\eta\pi_k \in \mathrm{End}(X_{r,s}) \cong \mathbb{C}$, hence there is a non-zero $\lambda \in \mathbb{C}$ such that $\eta\pi_k = \lambda\, \mathrm{id}_{X_{r,s}}$. Define $\chi = \frac{1}{\lambda}\pi_k$.

The proof of the second part of the lemma is analogous. □

Theorem 7.2.5. *For generic q, the functors $X_{m,n}$ are simple in $\mathfrak{Fun}^\circ(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$.*

Proof. Suppose $\mathcal{G}$ is a subobject of $X_{m,n}$. Then there is an exact sequence of functors

$$0 \longrightarrow \mathcal{G} \xrightarrow{\mu} X_{m,n} \xrightarrow{\epsilon} \mathcal{H} \longrightarrow 0 \; .$$

In particular, for each A, evaluating the above exact sequence of functors yields a short exact sequence of vector spaces:

$$0 \longrightarrow \mathcal{G}(A) \xrightarrow{\mu_A} X_{r,s}(A) \xrightarrow{\epsilon_A} \mathcal{H}(A) \longrightarrow 0 \; .$$

Suppose that $\mathcal{H}(\underline{r} \otimes \underline{s}^*) = 0$. Then, for $\phi \in X_{r,s}(A)$ the following diagram commutes:

$$\begin{tikzcd}
X_{r,s}(\underline{r} \otimes \underline{s}^*) \arrow[r, "\epsilon_{\underline{r}\otimes\underline{s}^*}"] \arrow[d, "\phi^*"'] & \mathcal{H}(\underline{r} \otimes \underline{s}^*) \arrow[d, "\mathcal{H}(\phi)"] \\
X_{r,s}(A) \arrow[r, "\epsilon_A"'] & \mathcal{H}(A)
\end{tikzcd} .$$

In particular,

$$\epsilon_A(\phi) = \epsilon_A \circ \phi^*(e_{r,s}) = \mathcal{H}(\phi) \circ \epsilon_{\underline{r}\otimes\underline{s}^*}(e_{r,s}) = 0 \, .$$

But cokernels in $\mathfrak{Fun}^\circ(\underline{\mathrm{TL}}, \underline{\mathrm{Vect}})$ are defined pointwise, whence $\mathcal{H}(A) = 0$ for every object A. Consequently, $\mathcal{G} \cong X_{r,s}$.

It remains to consider the case $\mathcal{H}(\underline{r} \otimes \underline{s}^*) \neq 0$. The following sequence is exact:

$$0 \longrightarrow \mathcal{G}(\underline{r} \otimes \underline{s}^*) \longrightarrow X_{r,s}(\underline{r} \otimes \underline{s}^*) \longrightarrow \mathcal{H}(\underline{r} \otimes \underline{s}^*) \longrightarrow 0 \, .$$

In particular,

$$X_{r,s}(\underline{r} \otimes \underline{s}^*) \cong \{\phi \colon \underline{r} \otimes \underline{s}^* \to \underline{r} \otimes \underline{s}^* \ : \ \phi e_{r,s} = \phi\} \cong \mathbb{C} \, .$$

Now, $\mathcal{H}(\underline{r} \otimes \underline{s}^*) \neq 0$ implies that $\mathcal{G}(\underline{r} \otimes \underline{s}^*) = 0$. Suppose that $\mathcal{G}(A) \neq 0$. Then there exists $v \in \mathcal{G}(A)$ such that $\mu_A(v) \neq 0$, since $\mathcal{G}$ is a kernel. Such a $\mu_A(v)$ is a morphism $A \to \underline{r} \otimes \underline{s}^*$ satisfying $\mu_A(v) e_{r,s} = \mu_A(v)$.

Since q is generic, Lemma 7.2.4 ensures that there is a natural transformation $\iota_* \colon X_{r,s} \to \yen(A)$ such that $0 \neq \iota\mu_A(v)$. The following diagram is commutative:

$$\begin{tikzcd}
\mathcal{G}(A) \arrow[r, "\mu_A"] \arrow[d, "\mathcal{G}(\iota)"'] & X_{r,s}(A) \arrow[d, "\iota^*"] \\
\mathcal{G}(\underline{r} \otimes \underline{s}^*) \arrow[r, "\mu_{\underline{r}\otimes\underline{s}^*}"'] & X_{r,s}(\underline{r} \otimes \underline{s}^*)
\end{tikzcd} .$$

For $v \in \mathcal{G}(A)$ fixed earlier, this yields

$$0 \neq \iota\mu_A(v) = \iota^* \circ \mu_A(v) = \mu_{\underline{r}\otimes\underline{s}^*} \circ \mathcal{G}(\iota)(v) = 0 \ ,$$

since $\mathcal{G}(\underline{r} \otimes \underline{s}^*) = 0$.

In conclusion, if $\mathcal{H}(\underline{r} \otimes \underline{s}^*) = 0$ then $\mathcal{G} \cong X_{r,s}$ and if $\mathcal{H}(\underline{r} \otimes \underline{s}^*) \neq 0$ then $\mathcal{G} = 0$. Hence $X_{r,s}$ is a simple object in $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$. □

Corollary 7.2.6. *For generic q, the abelian subcategory of $\mathfrak{Fun}^\circ(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$ generated by $\yen(\underline{Fus}_{\mathfrak{sl}_3})$ is semisimple.*

Let $\mathfrak{F} \in \mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$. Then $\mathrm{Hom}(\mathfrak{F}, \yen(A))$ is naturally an $\mathrm{End}(A)$ module. Henceforth, natural transformations will compose left to right to coincide with composition in $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$.

Proposition 7.2.7. *Suppose that $[k] \neq 0$ for every $1 \leq k \leq p+1$. Let A be an object of degree p. Then the* $\mathrm{End}(A)$*-module* $\mathrm{Hom}(X_{r,s}, \yen(A))$ *is simple.*

Proof. Let η and χ be non-zero natural transformations $X_{r,s} \to \yen(A)$. By Lemma 7.2.4 there exists a natural transformation $\phi \in \mathrm{Hom}(\yen(A), X_{r,s})$ satisfying $\eta\phi = \mathrm{id}_{X_{r,s}}$. Define $\psi = \phi\chi \in \mathrm{End}(\yen(A))$. Then $\psi = f_*$ for some morphism $f \in \mathrm{End}(A)$ and

$$\eta \cdot f = \eta\psi = \chi \ .$$

Consequently, every non-zero natural transformation $\eta \in \mathrm{Hom}(X_{r,s}, \yen(A))$ generates $\mathrm{Hom}(X_{r,s}, \yen(A))$ as an $\mathrm{End}(A)$-module. Hence, $\mathrm{Hom}(X_{r,s}, \yen(A))$ is simple. □

Define the functor $X_{r,s} \otimes V$ as

$$X_{r,s} \otimes V = (\yen(\underline{r} \otimes \underline{s}^*) \otimes V)^{e_{r,s}\otimes 1} \ .$$

Then it is easily shown, with the aid of Propositions 5.3.5 and 5.3.7, that

$$\begin{aligned} \mathrm{Hom}(\yen(A), X_{r,s} \otimes V) &\cong (e_{r,s} \otimes 1)_* \mathrm{Hom}(\yen(A), \yen(\underline{r} \otimes \underline{s}^*) \otimes V) \\ &\cong (e_{r,s})_* \mathrm{Hom}(A, \underline{r} \otimes \underline{s}^*) \otimes V \ , \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}(X_{r,s} \otimes V, \yen(A)) &\cong (e_{r,s} \otimes 1)^* \mathrm{Hom}(\yen(\underline{r} \otimes \underline{s}^*) \otimes V, \yen(A)) \\ &\cong \mathrm{Hom}(V, (e_{r,s})^* \mathrm{Hom}(\underline{r} \otimes \underline{s}^*, A)) \ . \end{aligned}$$

It is now possible to present a canonical semisimple decomposition for $\yen(A)$ as a Temperley-Lieb module, provided $\deg(A) = p$ is sufficiently small.

Theorem 7.2.8. *Suppose that $[k]_q \neq 0$ for every $1 \leq k \leq p+1$ and let A be an object of degree p. Then, as an* $\mathrm{End}(A)$*-module, the object $\yen(A)$ is semisimple and has a direct sum decomposition*

$$\yen(A) \cong \bigoplus_{\substack{r,s \\ r+s \leq p}} X_{r,s} \otimes_{\mathbb{C}} \mathrm{Hom}(X_{r,s}, \yen(A)) .$$

Proof. Consider the natural map

$$\chi\colon \bigoplus_{\substack{r,s \\ r+s \leq p}} X_{r,s} \otimes_{\mathbb{C}} \mathrm{Hom}(X_{r,s}, \yen(A)) \longrightarrow \yen(A)$$

defined as the direct sum of the evaluation maps. This is clearly a morphism of $\mathrm{End}(A)$-modules. It remains to show that it is an isomorphism. Choose a direct sum decomposition for $\yen(A)$ of the form

$$\yen(A) \cong \bigoplus_k X_{m_k,n_k} ,$$

including the inclusion maps $(\iota_k)_*$ and the projection maps $(\pi_k)_*$. For an object $B \in \underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$, define the map

$$\begin{aligned} \theta_B\colon (\yen(A))(B) &\longrightarrow \bigoplus_{\substack{r,s \\ r+s \leq p}} X_{r,s}(B) \otimes \mathrm{Hom}(X_{r,s}, \yen(A)) \\ \left(B \xrightarrow{f} A\right) &\longmapsto \sum_k f\pi_k \otimes (\iota_k)_* . \end{aligned}$$

Then $\chi \circ \theta = \mathrm{id}_{\yen(A)}$. In particular, θ_B is injective for every B. It remains to show that θ_B is surjective. Define $I(r,s)$ to be the subset of indices k such that $(\iota_k)_*$ has codomain $X_{r,s}$. Then $\{(\iota_k)_*\}_{k \in I(r,s)}$ is a spanning set for $\mathrm{Hom}(X_{r,s}, \yen(A))$. Explicitly, for $\eta \in \mathrm{Hom}(X_{r,s}, \yen(A))$,

$$\eta = \eta \, \mathrm{id}_{\yen(A)} = \sum_k \eta(\pi_k)_*(\iota_k)_* .$$

However, $\eta(\pi_k)_* = 0$ whenever $k \notin I(r,s)$ and $\eta(\pi_k)_* = \lambda_k \, \mathrm{id}_{X_{r,s}}$ whenever $k \in I(r,s)$. Hence

$$\eta = \sum_{k \in I(r,s)} \lambda_k (\iota_k)_* .$$

Let $f \in X_{r,s}(B)$. Then it is sufficient to show that $f \otimes (\iota_k)_* \in \mathrm{Im}(\theta_B)$ for each $k \in I(r,s)$. Consider $(\iota_k)_*(f) = f\iota_k \in (\yen(A))(B)$. Then

$$\theta_B(f\iota_k) = \sum_j f\iota_k\pi_j \otimes (\iota_j)_* = f \otimes (\iota_k)_* \ ,$$

as required. $\square$

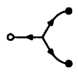
The analysis for singular q will be left until later.

7.2.2 A formal Koszul pair for generic q

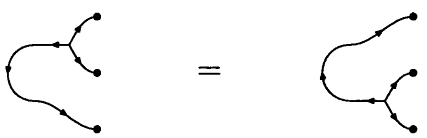
It will be shown that, as in the Temperley-Lieb case, there is a formal algebra-coalgebra pair in the functor category.

If q is generic then Theorem 7.2.1 constructs a sequence of bricks $X_{0,0}, X_{1,0}, X_{2,0}, \ldots$. If q is singular then there is a smallest natural number $h > 0$ such that $[h] = 0$ and Theorem 7.2.1 (together with Remark 7.2.2) builds precisely h bricks $X_{0,0}, \ldots, X_{h-1,0}$. In this case, define $X_{k,0}$ to be the zero functor for all $k \geq h$. The $X_{k,0}$ will be the graded pieces of a formal algebra in $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}_{\mathfrak{sl}_3}}, \underline{\mathrm{Vect}})$. Precisely as in the Temperley-Lieb case (Section 5.4.3), $X = \bigoplus_{p=0}^{\infty} X_{p,0}$ with the multiplication maps $\mu_{i,j} = (e_{i+j,0})_*\colon X_{i,0} \otimes X_{j,0} \longrightarrow X_{i+j,0}$ is an associative algebra.

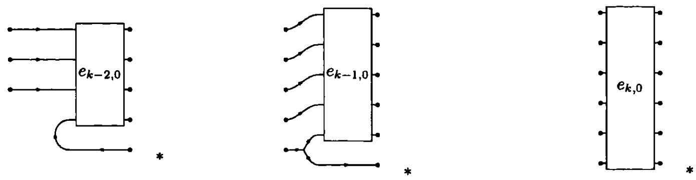
Define also the formal coalgebra $A = X_{0,0} \oplus X_{1,0} \oplus X_{0,1} \oplus X_{0,0}$, where the non-trivial components of the comultiplication are given by the operators

 $: A_2 \longrightarrow A_1 \otimes A_1$,  $: A_3 \longrightarrow A_1 \otimes A_2$,  $: A_3 \longrightarrow A_2 \otimes A_1$.

Coassociativity is easy to check and depends crucially on the relation



Theorem 7.2.9. *Suppose that $[k] \neq 0$ for every $1 \leq k \leq n$. Then the following sequences are exact:*



$$0 \longrightarrow X_{k-3,0} \otimes A_3 \longrightarrow X_{k-2,0} \otimes A_2 \longrightarrow X_{k-1,0} \otimes A_1 \longrightarrow X_{k,0} \longrightarrow 0\,.$$

Proof. The sequence is composed of the following two short sequences spliced together:

$$0 \longrightarrow X_{k-3,0} \otimes A_3 \longrightarrow X_{k-2,0} \otimes A_2 \longrightarrow X_{k-2,1} \longrightarrow 0$$

and

$$0 \longrightarrow X_{k-2,1} \longrightarrow X_{k-1,0} \otimes A_1 \longrightarrow X_{k,0} \longrightarrow 0\,.$$

These sequences are split exact by construction. □

Corollary 7.2.10. *For generic q, the pair (X, A) is Koszul.*

7.3 Graph representations

As in the Temperley-Lieb case, the objects of interest will be graph representations of the category $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$.

7.3.1 Definition and extensions

Definition 7.3.1. *A* GRAPH REPRESENTATION *of the category $\underline{Fus}_{\mathfrak{sl}_3}$ is a quiver Q together with a $\mathbb{C}$-linear (strict) monoidal functor $\mathfrak{F}\colon \underline{Fus}_{\mathfrak{sl}_3} \to (\mathbb{C}Q)_0\text{-}\underline{mod}\text{-}(\mathbb{C}Q)_0$ satisfying*

$$\begin{aligned} \mathfrak{F}(\underline{1}) &= (\mathbb{C}Q)_1\,, \\ \mathfrak{F}(\underline{1}^*) &= (\mathbb{C}Q^\circ)_1\,, \end{aligned}$$

where Q° denotes the dual (or opposite) quiver to Q.

Let A be an object in $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. Then A is a string of black and white dots. Fix a graph representation $\mathfrak{F}$ on the quiver Q and denote by $\overline{Q}$ the double of the quiver Q. A basis for $\mathfrak{F}(A)$ is the set of paths of length $\deg(A)$ in $\overline{Q}$ whose i^{th} arrow belongs to Q (resp. Q°) if the corresponding dot is black (resp. white).

Thus a graph representation is in particular a collection of operators on the path algebra of the doubled quiver Q satisfying certain relations. The construction of a graph representation for a given quiver Q is non-trivial and in practice graph representations are built using the quiver and some combinatorial data for certain configurations of triangles in the quiver. The construction of graph representations from this cell data on a quiver is discussed in Appendix B.

Let $\mathfrak{F}$ be a graph representation of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ and extend it in the usual way to $¥(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3})$ and the other functors occuring in the analysis of the category $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. For an object $A \in \underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$, a graph representation $\mathfrak{F}$ defines a representation of the algebra $\mathrm{End}(A)$ on $\mathfrak{F}(A) \subset \mathbb{C}\overline{Q}$. If q is generic, Theorem 7.2.8 ensures that these representations are semisimple and have the decomposition formula

$$\mathfrak{F}(A) \cong \bigoplus_{m,n} \mathfrak{F}(X_{m,n}) \otimes_{\mathbb{C}} \mathrm{Hom}(X_{m,n}, ¥(A)) ,$$

where for each (m,n), the object $\mathfrak{F}(X_{m,n})$ is the multiplicity in $\mathfrak{F}(A)$ of the simple $\mathrm{End}(A)$-module $\mathrm{Hom}(X_{m,n}, ¥(A))$. The $\mathfrak{F}(X_{m,n})$ carry the $(\mathbb{C}Q)_0$-bimodule structure on $\mathfrak{F}(A)$ and for each pair $i,j \in Q_0$, the direct summand $e_i\mathfrak{F}(X_{m,n})e_j$ must have non-negative dimension. Notice that, since $\mathfrak{F}$ is a $\mathbb{C}$-linear monoidal functor, the fusion rules for the simple objects $X_{m,n}$ are preserved, that is,

$$\begin{aligned}
\mathfrak{F}(X_{m,n}) \otimes_{(\mathbb{C}Q)_0} \mathfrak{F}(X_{1,0}) &\cong \mathfrak{F}(X_{m,n-1}) \oplus \mathfrak{F}(X_{m-1,n+1}) \oplus \mathfrak{F}(X_{m+1,n}) , \\
\mathfrak{F}(X_{m,n}) \otimes_{(\mathbb{C}Q)_0} \mathfrak{F}(X_{0,1}) &\cong \mathfrak{F}(X_{m-1,n}) \oplus \mathfrak{F}(X_{m+1,n-1}) \oplus \mathfrak{F}(X_{m,n+1}) ,
\end{aligned}$$

with the usual convention that if an index is negative then the corresponding summand is taken to be the zero object.

7.3.2 Koszul pairs from generic representations

Fix a quiver Q together with a graph representation $\mathfrak{F}$ of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. Recall that for singular q, the functor $X_{p,0}$ is defined to be the zero functor whenever $p \geq h$. Define

$$
\begin{aligned}
\Sigma &= \bigoplus_{p=0}^{\infty} \mathfrak{F}(X_{p,0}) \,, \\
\Lambda &= \mathfrak{F}(X_{0,0}) \oplus \mathfrak{F}(X_{1,0}) \oplus \mathfrak{F}(X_{0,1}) \oplus \mathfrak{F}(X_{0,0}) \,,
\end{aligned}
$$

effectively the image under $\mathfrak{F}$ of the formal algebra and the formal coalgebra defined in Section 7.2.2. Since Σ and Λ inherit the algebra and coalgebra structure respectively, Corollary 7.2.10 immediately yields:

Corollary 7.3.2. *If q is generic then (Σ, Λ) is a Koszul pair.*

Consider the two-sided ideal $\mathfrak{J} \subset \mathbb{C}Q$ generated by $\mathrm{Im}(\prec)$. Define the quotient algebra Π via the short exact sequence

$$
0 \longrightarrow \mathfrak{J} \longrightarrow \mathbb{C}Q \longrightarrow \Pi \longrightarrow 0 \,.
$$

Notice that Π is a homogeneous ideal, hence Π inherits the grading from $\mathbb{C}Q$. It will be shown (but see Remark 5.5.4) that the graded pieces of Π can be identified with the spaces $\Sigma_p = \mathfrak{F}(X_{p,0}) \subset \mathbb{C}Q$ and that, for generic q, the algebras Π and Σ are isomorphic. In particular, (Π, Λ) is a Koszul pair.

There is an obvious candidate for such an isomorphism $\Sigma \to \Pi$, given by the composite of the inclusion map $j \colon \Sigma \to \mathbb{C}Q$ and the quotient map $\pi \colon \mathbb{C}Q \to \Pi$.

Recall that $\Pi^{(\leq n)}$ denotes the truncation of the algebra Π after degree n. The following proposition is proved in an identical way to Proposition 5.5.6:

Proposition 7.3.3. *Suppose that $[k] \neq 0$ for every $1 \leq k \leq n$. Then the induced map $\Sigma \to \Pi^{(\leq n)}$ is an algebra homomorphism.*

Corollary 7.3.4. *For generic q, the map $\pi \circ j \colon \Sigma \to \Pi$ is an algebra homomorphism.*

As in the Temperley-Lieb case, there is an exact sequence

$$
\Pi_{n-2} \otimes \Lambda_2 \longrightarrow \Pi_{n-1} \otimes \Lambda_1 \longrightarrow \Pi_n \longrightarrow 0 \,,
$$

where the maps are precisely the Koszul differential on $\Pi_\bullet \otimes \Lambda_\bullet$.

In particular, provided $[k] \neq 0$ for every $2 \leq k \leq n$, the algebra homomorphism $\Sigma \to \Pi^{(\leq n)}$ induces a map of exact sequences

$$\begin{tikzcd}
\Sigma_{k-2} \otimes \Lambda_2 \arrow[r] \arrow[d] & \Sigma_{k-1} \otimes \Lambda_1 \arrow[r] \arrow[d] & \Sigma_k \arrow[r] \arrow[d] & 0 \\
\Pi_{k-2} \otimes \Lambda_2 \arrow[r] & \Pi_{k-1} \otimes \Lambda_1 \arrow[r] & \Pi_k \arrow[r] & 0 .
\end{tikzcd}$$

Theorem 7.3.5. *Suppose that $[k] \neq 0$ for every $1 \leq k \leq n$. Then the map $\Sigma_n \to \Pi_n$ is an isomorphism.*

Proof. The algebras coincide in degree 0 and 1 and the result now follows from the 5-lemma by induction. □

Corollary 7.3.6. *Suppose that $[k] \neq 0$ for every $1 \leq k \leq n$. Then the truncated algebras $\Sigma^{(\leq n)}$ and $\Pi^{(\leq n)}$ are isomorphic.*

Corollary 7.3.7. *For generic q, the algebras Σ and Π are isomorphic.*

Corollary 7.3.8. *Suppose that $[k] \neq 0$ for every $1 \leq k \leq n \leq 3$. Then the following sequence is exact:*

$$0 \longrightarrow \Pi_{n-3} \otimes \Lambda_3 \longrightarrow \Pi_{n-2} \otimes \Lambda_2 \longrightarrow \Pi_{n-1} \otimes \Lambda_1 \longrightarrow \Pi_n \longrightarrow 0 .$$

Corollary 7.3.9. *For generic q, the pair (Π, Λ) is Koszul.*

7.4 The reduced category

It will be shown that if $\mathfrak{F}$ is a graph representation factoring through a suitable tensor ideal in $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$, then the algebra-coalgebra pair (Π, Λ) is almost Koszul.

7.4.1 Defining the reduced category

Recall that a subcategory $\mathfrak{I}$ of a monoidal category $\mathcal{C}$ is said to be a tensor ideal if it is closed under arbitrary compositions and tensor products.

First, construct a trace using the pairing and copairing operators. Let A be an object in $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. There are well-defined maps

$$\begin{array}{rcl} \mathrm{End}(A \otimes \underline{1}) & \longrightarrow & \mathrm{End}(A) \\ f & \longmapsto & (\mathrm{id}_A \otimes \text{C})(f \otimes 1)(\mathrm{id}_A \otimes \text{Ↄ}) \end{array}$$

and

$$\begin{array}{rcl} \mathrm{End}(A \otimes \underline{1}^*) & \longrightarrow & \mathrm{End}(A) \\ f & \longmapsto & (\mathrm{id}_A \otimes \subset)(f \otimes 1)(\mathrm{id}_A \otimes \supset)\,. \end{array}$$

Every object is a tensor product of $\underline{1}$s and $\underline{1}^*$s, hence iteratively applying the above maps and composing with the identification $\mathrm{End}(\underline{0}) \cong \mathbb{C}$ yields a collection of maps called the TRACE and denoted by $f \mapsto \mathrm{tr}(f)$. It will occasionally be convenient to think of $\mathrm{tr}(f)$ as a morphism in $\mathrm{End}(\underline{0})$.

Notice that it is by no means clear that $\mathrm{tr}(fg) = \mathrm{tr}(gf)$.

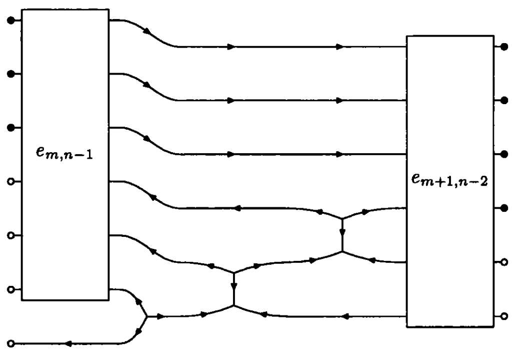
Lemma 7.4.1. *The idempotent $e_{m,n} \in \mathrm{End}(\underline{m} \otimes \underline{n}^*)$ has trace*

$$\mathrm{tr}(e_{m,n}) = \frac{[m+n+2][m+1][n+1]}{[2]}\,.$$

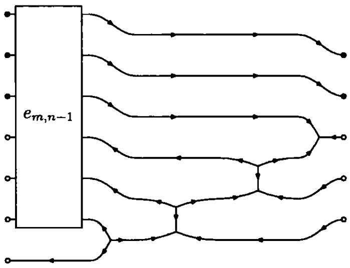
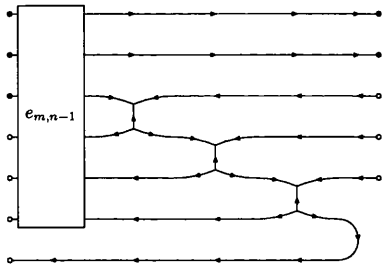
Proof. The formula is certainly correct for $e_{0,0}$, $e_{1,0}$ and $e_{0,1}$. Consider $e_{m,n}$ with $m+n \geq 1$ and suppose the result holds for all $e_{i,j}$ with $i+j \leq m+n-1$. Now

$$e_{m,n} \;=\; e_{m,n-1} \otimes \underline{1}^* - \frac{[n-1]}{[n]}\psi\psi^\circ - \frac{[m][m+n]}{[m+1][m+n+1]}\phi\phi^\circ\,,$$

where

$\psi \;=$ 

and

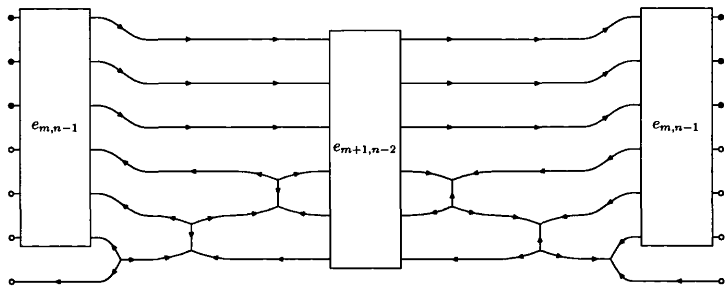
ϕ =  = 

Thus,

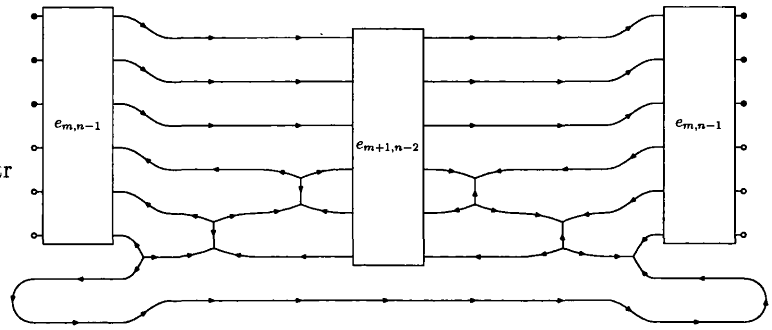
$$\operatorname{tr}(e_{m,n}) \;=\; \operatorname{tr}(e_{m,n-1}\otimes 1) - \frac{[n-1]}{[n]}\operatorname{tr}(\psi\psi^{\circ}) - \frac{[m][m+n]}{[m+1][m+n+1]}\operatorname{tr}(\phi\phi^{\circ})\,.$$

But $\operatorname{tr}(\phi\phi^{\circ}) = \operatorname{tr}(e_{m,n-1})$ and the calculation for $\operatorname{tr}(\psi\psi^{\circ})$ is as follows:

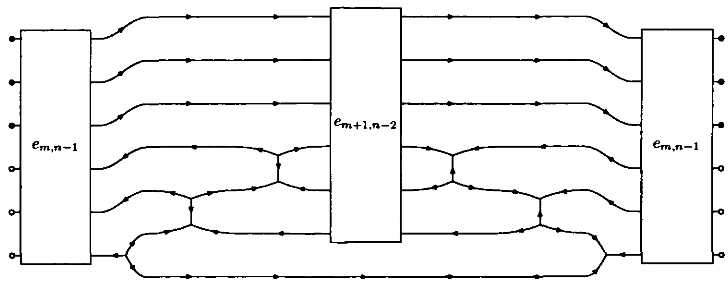
$\mathrm{tr}(\psi\psi^\circ) \quad = \quad \mathrm{tr}$



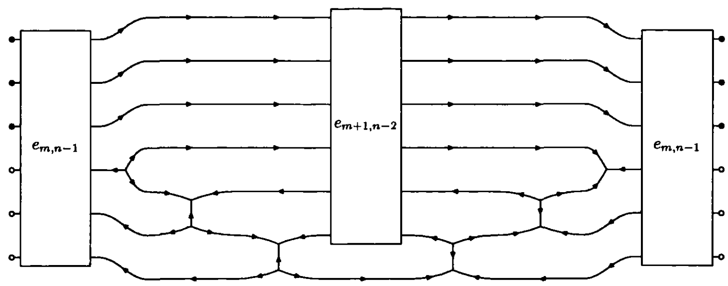
$= \quad \mathrm{tr}$



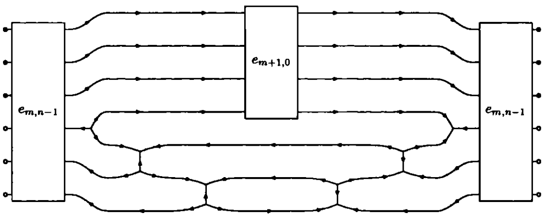
$= \quad \mathrm{tr}$

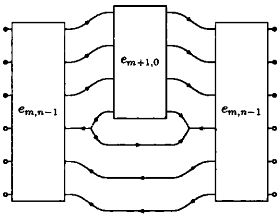


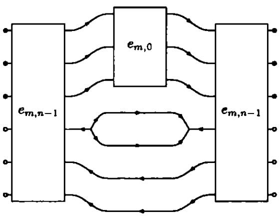
$= \quad \mathrm{tr}$

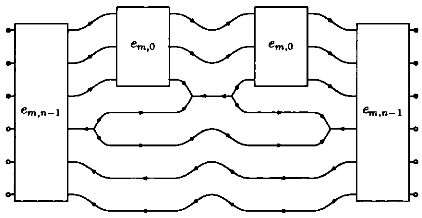


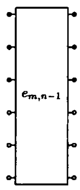
Now use the inductive formula for $e_{m+1,n-2}$ and the fact that there are no non-zero maps between $X_{m,n-1}$ and ¥$(\underline{m+3} \otimes \underline{n-4}^*)$, yielding

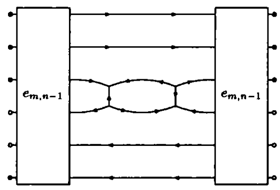
$\mathrm{tr}(\psi\psi^{\circ}) \;=\; \mathrm{tr}$ 

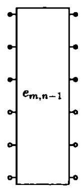
$= \; \mathrm{tr}$ 

$= \; \mathrm{tr}$ 

$- \; \frac{[m]}{[m+1]} \, \mathrm{tr}$ 

$= \; [2] \, \mathrm{tr}$ 

$- \; \frac{[m]}{[m+1]} \, \mathrm{tr}$ 

$= \; \frac{[m+2]}{[m+1]} \, \mathrm{tr}$ 

Hence,

$$
\begin{aligned}
\mathrm{tr}(e_{m,n}) &= \mathrm{tr}(e_{m,n-1} \otimes 1) - \frac{[n-1]}{[n]} \mathrm{tr}(\psi\psi^{\circ}) - \frac{[m][m+n]}{[m+1][m+n+1]} \mathrm{tr}(\phi\phi^{\circ}) \\
&= \left([3] - \frac{[m+2][n-1]}{[m+1][n]} - \frac{[m][m+n]}{[m+1][m+n+1]} \right) \mathrm{tr}(e_{m,n-1}) \, .
\end{aligned}
$$

Now using the induction hypothesis yields

$$\begin{aligned}
[2]\,\mathrm{tr}(e_{m,n}) &= \left([3] - \frac{[m+2][n-1]}{[m+1][n]} - \frac{[m][m+n]}{[m+1][m+n+1]}\right)[m+n+1][m+1][n] \\
&= [3][m+n+1][m+1][n] - [m+n+1][m+2][n-1] - [m+n][m][n] \\
&= [m+n+3][m+1][n] \\
&\quad - [m+n+1]\left([m+2][n-1] - [m+1][n]\right) \\
&\quad - [n]\left([m+n][m] - [m+n-1][m+1]\right) \ .
\end{aligned}$$

Recall the formula

$$[r][s] = \sum_{k=0}^{r} [r+s-2k+1] \ ,$$

to obtain

$$
\begin{aligned}
[2]\,\mathrm{tr}(e_{m,n}) &= \sum_{i=1}^{m+n+3}\sum_{j=1}^{m+1}[2m+2n-2i-2j+6] \\
&\quad - [m+n+1]\left(\sum_{j=1}^{m+2}[m+n-2j+2] - \sum_{j=1}^{m+1}[m+n-2j+2]\right) \\
&\quad - [n]\left(\sum_{i=1}^{m+n}[2m+n-2i+1] - \sum_{i=1}^{m+n-1}[2m+n-2i+1]\right) \\
&= \sum_{i=1}^{m+n+3}\sum_{j=1}^{m+1}[2m+2n-2i-2j+6] \\
&\quad - [m+n+1][-m+n-2] - [n][-n+1] \\
&= \sum_{i=1}^{m+n+2}\sum_{j=1}^{m+1}[2m+2n-2i-2j+6] + \sum_{j=1}^{m+1}[-2j] \\
&\quad - [m+n+1][-m+n-2] + [n][n-1] \\
&= [m+n+2][m+1][n+1] \\
&\quad - \sum_{j=1}^{m+1}[2j] - \sum_{i=1}^{m+n+1}[2n-2i] + \sum_{k=1}^{n}[2n-2k] \\
&= [m+n+2][m+1][n+1] - \sum_{j=1}^{m+1}[2j] + \sum_{i=n+1}^{m+n+1}[2i-2n] \\
&= [m+n+2][m+1][n+1]\,,
\end{aligned}
$$

as required. $\square$

Define the ideal of negligible morphisms, $\mathcal{N}\mathrm{eg}$, to be the tensor ideal given by

$$\mathcal{N}\mathrm{eg}(A,B) = \{f \in \mathrm{Hom}(A,B) \ :\ \mathrm{tr}(hfg) = 0 \quad \forall\, h \in \mathrm{Hom}(C,A) \text{ and } g \in \mathrm{Hom}(B,C)\}\,.$$

Notice that, as in the Temperley-Lieb case, every proper tensor ideal must be contained in $\mathcal{N}\mathrm{eg}$.

Proposition 7.4.2. *For generic q, the category $\underline{\mathit{Fus}}_{\mathfrak{sl}_3}$ contains no proper tensor ideals.*

Proof. The category $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$ is semisimple and for every simple object the identity map has non-zero trace. Let $\phi \in \mathrm{Hom}(A, B)$ be non-zero. Then there is a simple object $X_{m,n}$ and inclusion and projection maps, ι and π such that $0 \neq \iota\phi\pi \in \mathrm{End}(X_{m,n})$. Hence $\mathrm{tr}(\iota\phi\pi) \neq 0$. □

Proposition 7.4.3. *For singular q, the category $\mathfrak{Fun}^\circ(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$ contains at least one proper tensor ideal.*

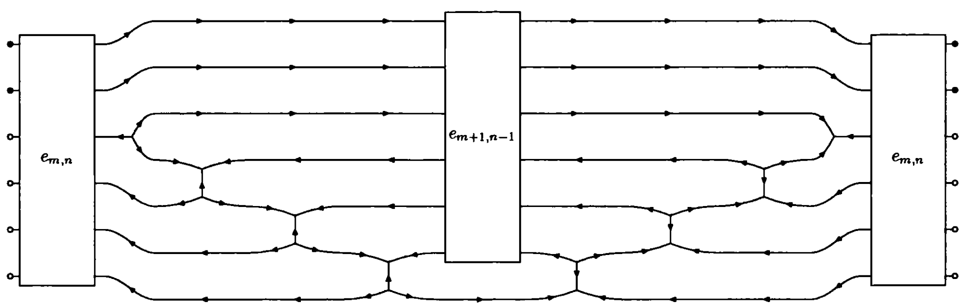
Proof. Let h denote the Coxeter number and consider the graph representation $\mathfrak{F}$ of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ constructed in Appendix B for the $A^{(h)}$ quiver. For $S = \left(\mathbb{C}A^{(h)}\right)_0$, denote by M_k, the decomposition matrix of the S-bimodule $\mathfrak{F}(X_{k,0})$. By Theorem 7.2.9, the matrices M_i form the initial portion of a solution to the SL_3 recurrence and Theorem 3.4.8 assures that $M_{h-2} = 0$. Hence $\mathfrak{F}(X_{h-2,0}) = 0$ and since $\mathfrak{F}$ is a non-trivial $\mathbb{C}$-linear monoidal functor, the ideal generated by $e_{h-2,0}$ must be a proper tensor ideal of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. □

Henceforth, suppose that q is singular and let $\mathcal{N} = \langle e_{h-2,0}\rangle$ denote the proper tensor ideal generated by $e_{h-2,0}$. Define $\mathcal{N}(A, B) = \mathcal{N} \cap \mathrm{Hom}(A, B)$.

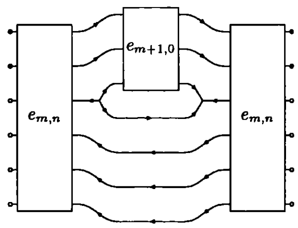
Theorem 7.4.4. *The proper tensor ideal $\mathcal{N} = \langle e_{h-2,0}\rangle$ contains $e_{m,n}$ for every $m+n = h-2$.*

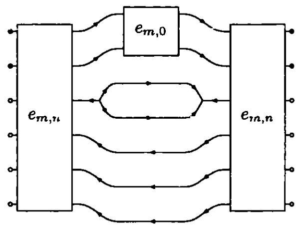
Proof. Suppose that $e_{m+1,n-1} \in \mathcal{N}$ and consider the following element of $\mathcal{N}$:

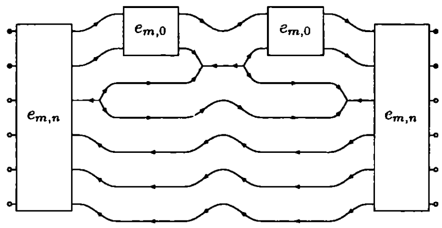
$\phi \quad =$

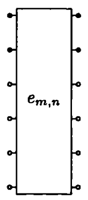


Using the inductive formula for $e_{m+1,n-1}$ and the fact that there are no non-zero maps from $X_{m,n}$ to $\mathfrak{Y}(\underline{m+1} \otimes \underline{n-1})$ now gives

$\phi =$ 

$=$ 

$- \frac{[m-1]}{[m]}$ 

$= \frac{[m+1]}{[m]}$ 

Finally, $[m+1] \neq 0$ since $m+1 \leq h-1$. Hence, $e_{m,n} \in \mathcal{N}$. $\square$

The REDUCED FUSION CATEGORY, $\underline{\mathrm{Fus}}^{\mathrm{red}}_{\mathfrak{sl}_3}$, is defined to be the categorical quotient of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ by the proper tensor ideal $\mathcal{N} = \langle e_{h-2,0} \rangle$. Denote Hom-sets in the reduced category by $\mathrm{Hom}_{\mathrm{red}}(A, B)$.

Remark 7.4.5. *The notation $\mathcal{N}$ is deliberately suggestive, since it will emerge shortly that for singular q, the tensor ideal $\mathcal{N}$ and $\mathcal{N}$eg coincide.*

The aim is to show that the analysis performed for the generic fusion category passes to the reduced category. In particular, quotienting by the tensor ideal $\mathcal{N}$ leaves, in an appropriate sense, a semisimple category. Moreover, graph representations of this category can be used to build almost Koszul algebra-coalgebra pairs.

Extend the quotient functor to the objects pertinent to the analysis of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ (as in the Temperley-Lieb case, Section 5.6.2) and denote the resulting functor by R. The monoidal structure can be pushed forward to make R a monoidal functor.

7.4.2 Analysing the reduced category

The analysis for the reduced fusion category is similar to that for the reduced Temperley-Lieb category. The first step is to identify a distinguished collection of bricks in the reduced category.

Lemma 7.4.6. *Consider the distinguished collection of bricks $\{X_{i,j} : i,j \in \mathbb{N} \cup \{0\}\ ,\ \ i+j \le h-3\}$ in $\mathfrak{Fun}^{\circ}(\underline{Fus}_{\mathfrak{sl}_3}, \underline{Vect})$ constructed by Theorem 7.2.1. Then $R(X_{i,j})$ is a brick for $i+j \le h-3$ and $\mathrm{Hom}_{\mathrm{red}}(R(X_{i,j}), R(X_{m,n})) = 0$ whenever $(i,j) \neq (m,n)$.*

Proof. By the construction in Theorem 7.2.1, $\mathrm{End}(X_{i,j}) \cong \mathbb{C}e_{i,j}$. But, $e_{i,j}$ has non-zero trace and thus has a non-zero image in the quotient category $\underline{\mathrm{Fus}}^{\mathrm{red}}_{\mathfrak{sl}_3}$, whence $\mathrm{End}(R(X_{i,j})) \cong \mathbb{C}r(e_{i,j}) \cong \mathbb{C}$. Furthermore, there are no non-zero maps between distinct $X_{i,j}$, thus the result also passes to the reduced objects $R(X_{i,j})$. $\square$

Identify the functor category $\mathfrak{Fun}^{\circ}(\underline{\mathrm{Fus}}^{\mathrm{red}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$ with the subcategory of $\mathfrak{Fun}^{\circ}(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$ consisting of the functors that vanish on $\mathcal{N}$. The bricks $R(X_{i,j})$ are thus identified with the quotient of the bricks $X_{i,j}$ by a proper subobject, $X^{\mathcal{N}}_{i,j}$.

Proposition 7.4.7. *For singular q and for $i+j \le h-3$, the functor $X_{i,j}$ has a strict subobject $X^{\mathcal{N}}_{i,j}$ defined by*

$$\begin{aligned} X^{\mathcal{N}}_{i,j}(A) &= \{\phi \in \mathrm{Hom}(A, \underline{i} \otimes \underline{j}^*) \ :\ \phi e_{i,j} = \phi\} = \mathcal{N}(A, \underline{i} \otimes \underline{j}^*) e_{i,j}\ , \\ X^{\mathcal{N}}_{i,j}(\phi) &= \phi^*\ . \end{aligned}$$

Proof. The functor $X^{\mathcal{N}}_{i,j}$ is well-defined since $\mathcal{N}$ is an ideal and is a subobject of $X_{i,j}$ via the natural inclusion maps. Furthermore, $e_{i,j} \in X_{i,j}(\underline{i} \otimes \underline{j}^*)$ but $e_{i,j}$ is not an element of $\mathcal{N}(\underline{i} \otimes \underline{j}^*, \underline{i} \otimes \underline{j}^*)$ because it has a non-zero trace. Hence $e_{i,j} \notin X^{\mathcal{N}}_{i,j}(\underline{i} \otimes \underline{j}^*)$. $\square$

Conjecture 7.4.8. *I believe that $X^{\mathcal{N}}_{i,j}$ is a proper subobject of $X_{i,j}$. For this, it remains to prove that $X^{\mathcal{N}}_{i,j}$ is not the zero functor. I have no direct proof of this. (I think the analogous argument to the Temperley-Lieb case should work, but this relied on a combinatoric understanding of the Temperley-Lieb diagrams that I am currently lacking in the $\mathfrak{sl}_3$ case.) This conjecture is stated for the sake of completeness only and has no bearing on the veracity of what follows.*

Corollary 7.4.9. *Let q be singular and let $i+j \le h-3$. Extend $R(X_{i,j})$ to a functor $\underline{Fus}_{\mathfrak{sl}_3} \to \underline{Vect}$ that vanishes on $\mathcal{N}$. Then $R(X_{i,j})$ is the cokernel of the inclusion map $X^{\mathcal{N}}_{i,j} \to X_{i,j}$.*

Proof. Cokernels in $\mathfrak{Fun}^{\circ}(\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$ are defined pointwise. Now

$$R(X_{i,j})(A) = \frac{\mathrm{Hom}(A, \underline{i} \otimes \underline{j}^*)}{\mathcal{N}(A, \underline{i} \otimes \underline{j}^*)} \cdot r(e_{i,j}) = \frac{\mathrm{Hom}(A, \underline{i} \otimes \underline{j}^*) e_{i,j}}{\mathcal{N}(A, \underline{i} \otimes \underline{j}^*) e_{i,j}} = \frac{X_{i,j}(A)}{X^{\mathcal{N}}_{i,j}(A)}\ ,$$

as required. $\square$

The functor R is additive and monoidal, hence the bricks $R(X_{i,j})$ satisfy a truncated fusion rule:

$$\begin{aligned} R(X_{i,j}) \otimes R(X_{1,0}) &\cong R(X_{i,j-1}) \oplus R(X_{i-1,j+1}) \oplus R(X_{i+1,j}) , \\ R(X_{i,j}) \otimes R(X_{0,1}) &\cong R(X_{i-1,j}) \oplus R(X_{i+1,j-1}) \oplus R(X_{i,j+1}) , \end{aligned}$$

where $0 \leq i+j \leq h-3$ and the corresponding brick $R(X_{i,j})$ is zero if $i+j = h-2$ and is defined to be zero if either of the indices are negative.

In particular, it is now a simple induction to prove that $\{R(X_{i,j}) : 0 \leq i+j \leq h-3\}$ is a complete set of indecomposable summands for the additive subcategory of $\mathfrak{Fun}^\circ(\underline{\mathrm{Fus}}^{\mathrm{red}}_{\mathfrak{sl}_3}, \underline{\mathrm{Vect}})$ generated by $\yen(\underline{\mathrm{Fus}}^{\mathrm{red}}_{\mathfrak{sl}_3})$.

Subsequent analysis of the reduced category $\underline{\mathrm{Fus}}^{\mathrm{red}}_{\mathfrak{sl}_3}$ can now proceed in an identical fashion to that for $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ for generic values of q. In particular,

Theorem 7.4.10. *The functors $R(X_{i,j})$ are simple in $\mathfrak{Fun}^\circ(\underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3}, \underline{Vect})$.*

Corollary 7.4.11. *The abelian subcategory of $\mathfrak{Fun}^\circ(\underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3}, \underline{Vect})$ generated by $\yen(\underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3})$ is semisimple.*

Theorem 7.4.12. *For each object A, the $\mathrm{End}_{\mathrm{red}}(A)$-module $\mathrm{Hom}_{\mathrm{red}}(R(X_{i,j}), \yen(A))$ is simple.*

Theorem 7.4.13. *For every $\underline{A} \in \underline{Fus}^{\mathrm{red}}_{\mathfrak{sl}_3}$, the object $\yen(A)$ is semisimple as an $\mathrm{End}_{\mathrm{red}}(A)$-module and has a canonical direct sum decomposition*

$$\yen(A) \cong \bigoplus_{0 \leq i+j \leq h-3} R(X_{i,j}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{red}}(R(X_{i,j}), \yen(A)) .$$

To conclude this section, another corollary of Theorem 7.4.10 is that the tensor ideals $\mathcal{N}\mathrm{eg}$ and $\mathcal{N}$ must coincide.

Corollary 7.4.14. $\mathcal{N}\mathrm{eg} = \mathcal{N}$.

Proof. The trace function can be pushed forward to the reduced category and the proof is now analogous to that for Proposition 7.4.2. But ideals in the reduced category are in one-to-one correspondence with ideals of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ containing $\mathcal{N}$. $\square$

7.4.3 Almost Koszul pairs from reduced representations

Let q be singular. A REDUCED GRAPH REPRESENTATION is a graph representation of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ factoring through the proper tensor ideal $\mathcal{N}\mathrm{eg}$.

Proposition 7.4.15. *Let Γ be a Di Francesco-Zuber $\widehat{\mathfrak{sl}}_3$ graph and assume that Γ supports a graph representation $\mathfrak{F}$ of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$. Then $\mathfrak{F}$ is a reduced representation.*

Proof. Denote the decomposition matrix of $\mathfrak{F}(X_{i,0})$ by M_i. By Theorem 7.2.9 the M_i generate an initial segment of a solution to the SL_3 recurrence. This solution is finite and satisfies $M_{h-2} = 0$ by Corollary 3.4.9. In particular, $X_{h-2,0}$ is the zero object and hence the graph representation $\mathfrak{F}$ factors through the tensor ideal generated by $e_{h-2,0}$. Thus, $\mathfrak{F}$ is indeed a reduced representation. □

Extend the reduced graph representation $\mathfrak{F}$ as in Section 5.5.2. Then for each A, the $\mathrm{End}_{\mathrm{red}}(A)$-module $\mathfrak{F}(A)$ has a canonical semisimple decomposition

$$\mathfrak{F}(A) \cong \bigoplus_{0\leq i+j\leq h-3} \mathfrak{F}(X_{i,j}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{red}}(R(X_{i,j}), \yen(A)) \ .$$

To conclude the chapter, it is shown that reduced graph representations define an almost Koszul algebra-coalgebra pair.

Recall that for singular q, the functors $X_{p,0}$ with $p \geq h$ are defined to be the zero functor. Define

$$\begin{aligned} \widetilde{X} &= \bigoplus_p R(X_{p,0}) \ , \\ \widetilde{A} &= R(X_{0,0}) \oplus R(X_{1,0}) \oplus R(X_{0,1}) \oplus R(X_{0,0}) \ , \end{aligned}$$

and equip $\widetilde{X}$ and $\widetilde{A}$ with the induced algebra and coalgebra structures from X and A respectively.

Theorem 7.4.16. *The formal algebra-coalgebra pair $(\widetilde{X}, \widetilde{A})$ is almost Koszul.*

Proof. Theorem 7.2.9 and the fact that R is additive and monoidal ensure that, for every $3 \leq k \leq h-1$, the following sequences are exact,

$$0 \longrightarrow \widetilde{X}_{k-3} \otimes \widetilde{A}_3 \longrightarrow \widetilde{X}_{k-2} \otimes \widetilde{A}_2 \longrightarrow \widetilde{X}_{k-1} \otimes \widetilde{A}_1 \longrightarrow \widetilde{X}_k \longrightarrow 0 \ ,$$

where the maps are given by the Koszul differential on $\widetilde{X}_\bullet \otimes \widetilde{A}_\bullet$. In particular, $\widetilde{X}_{h-2,0} = R(X_{h-2,0}) = 0 = R(X_{h-1,0}) = \widetilde{X}_{h-1,0}$. Thus the exact sequence for $k = h-2$

degenerates to give the short exact sequence

$$0 \longrightarrow \widetilde{X}_{h-5} \otimes \widetilde{A}_3 \longrightarrow \widetilde{X}_{h-4} \otimes \widetilde{A}_2 \longrightarrow \widetilde{X}_{h-3} \otimes \widetilde{A}_1 \longrightarrow 0$$

and the exact sequence for $k = h - 1$ degenerates to give $\widetilde{X}_{h-4} \otimes \widetilde{A}_3 \cong \widetilde{X}_{h-3} \otimes \widetilde{A}_2$, as required. $\square$

Let $\mathfrak{F}$ be a reduced graph representation and recall the definitions from Section 7.3.2:

$$\begin{aligned} \Sigma &= \bigoplus_p \mathfrak{F}(X_{p,0}) \, , \\ \Lambda &= \mathfrak{F}(X_{0,0}) \oplus \mathfrak{F}(X_{1,0}) \oplus \mathfrak{F}(X_{0,1}) \oplus \mathfrak{F}(X_{0,0}) \, , \\ \Pi &= \frac{\mathbb{C}Q}{\mathfrak{J}} \, , \text{ where } \mathfrak{J} \text{ is the two-sided ideal in } \mathbb{C}Q \text{ generated by } \mathrm{Im}(\prec). \end{aligned}$$

Since $\mathfrak{F}$ is a reduced representation, there is an immediate corollary to Theorem 7.4.16:

Corollary 7.4.17. *The algebra-coalgebra pair* (Σ, Λ) *is almost Koszul.*

Corollary 7.3.6 assures that $\Sigma \cong \Pi^{(\leq h-1)}$. However, $\mathfrak{F}$ is a reduced representation, hence $\Pi_{h-1} \cong \Sigma_{h-1} = \mathfrak{F}(X_{h-1,0}) = 0$. But Π is generated by Π_1, so $\Pi_k = 0$ whenever $k \geq h - 1$. Thus $\Pi = \Pi^{(\leq h-1)}$. Consequently (see also Remark 5.6.15):

Proposition 7.4.18. *The algebras* Σ *and* Π *are isomorphic.*

Corollary 7.4.19. *The algebra-coalgebra pair* (Π, Λ) *is almost Koszul.*

In particular, every graph representation of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ supported by an $\widehat{\mathfrak{sl}}_3$ graph Γ , defines a new and potentially interesting almost Koszul algebra-coalgebra pair (Π, Λ), where Π is the quotient of the path algebra $\mathbb{C}\Gamma$ by the ideal generated by $\mathrm{Im}(\prec)$.

Chapter 8

Conclusion and further work

This thesis furnishes a deeper understanding of *why* the preprojective algebras of the Dynkin quivers are almost Koszul and (provided suitable weights exist as Ocneanu claims) proves that all the quivers associated to $c < 2$ rational boundary conformal field theories possess almost Koszul algebras, greatly increasing the number of known examples. The key message is that what happens is *not* simply a failure of Koszulity. Both Koszul and almost Koszul preprojective algebras arise in precisely the same way (via representations of the Temperley-Lieb category) but, if anything, the Dynkin preprojective algebras are a little more special, factoring through an ideal in the Temperley-Lieb category that is trivial in the generic case.

The Temperley-Lieb and $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ categories are, furthermore, candidates for understanding (or ignoring) continuum limits of the Pasquier models. In particular, much (if not all) of the content of the $c < 1$ rational boundary conformal field theories should arise from graph representations (with additional structure) on the Temperley-Lieb category.

The diagram-theoretic approach to constructing the preprojective algebras and their analogues, permitted a universal and straightforward proof of (almost) Koszulity. It is worth investigating the possibility that certain other properties of the preprojective algebras (self-injective, symmetric, (stably) Calabi-Yau) might have universal diagrammatic proofs. Moreover, similar properties could be discerned for their $c < 2$ analogues and their relevance to conformal field theory analysed.

It seems that the Dynkin preprojective algebras and their analogues play an important role in rational conformal field theory. Almost Koszulity is closely tied to rationality and the decomposition matrices of the graded pieces of these algebras arise in the partition function of the cylinder. Coquereaux [CHST06, CG05] explicitly constructs the product on the preprojective algebra and defines two bialgebra structures on the space of graded endomorphisms of essential paths, $\bigoplus_n \mathrm{End}_{\mathbb{C}}(\mathrm{EssPath}_n)$, which are related to *Ocneanu's algebra of quantum symmetries* for the graph. It would be interesting to understand these constructions from the functorial point of view in this thesis, as well as establishing how they relate to the conformal field theory in the limit.

Coquereaux [Coq02] also presents a method for constructing the modular invariant function for the torus using essential paths. It should be possible to mimic the analysis performed in this thesis for the *affine* Temperley-Lieb algebras with the aim of constructing the coefficients of the modular invariant functions as multiplicity spaces. Coquereaux's construction may be important in identifying a Temperley-Lieb bimodule structure, or may be a discrete analogue of *Cardy's equation*, relating two different ways of computing the partition function on a cylinder (see Figure 8-1).

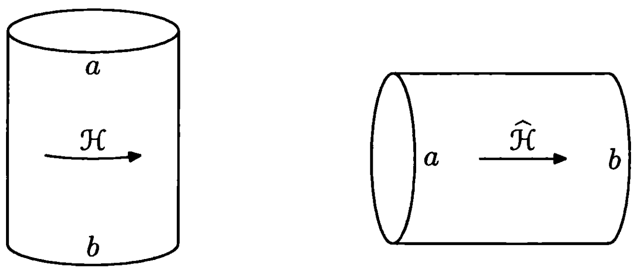


Figure 8-1: The one-loop evolution of an open string versus the tree approximation of the propagation of a closed string.

There is also the possibility of a homological interpretation of the proceeding paragraph. The preprojective algebra can be interpreted as the *Yoneda cohomology* of the trivial extension algebra and some preliminary computations suggest that the multiplicities arising in the modular invariant function are related to the *Hochschild cohomology* of the trivial extension algebra. Coquereaux's construction and Cardy's equation might thus be interpreted as a relation between Yoneda and Hochschild cohomology.

This thesis provides a way of constructing new and interesting examples of almost Koszul algebras and establishes a framework in which their role in conformal field theory might be elicited. It is my hope that it serves as an invitation to a better and deeper understanding of the relationship between conformal field theory and quiver algebras.

Appendix A

The scalar recurrence

This appendix provides some analysis of the scalar recurrences obtained by diagonalising the SL_n recurrence from Chapter 3.

A.1 The initial value problem

Let $p_0, \ldots, p_n$ be complex numbers and suppose that $p_0, p_n \neq 0$. Consider the degree n recurrence

$$\sum_{i=0}^{n} p_i x_{t+i} = 0 \tag{A.1}$$

together with the initial condition

$$x_0 = 1 \quad \text{and} \quad x_{-t} = 0 \ \forall t = 1, \ldots, n-1 \ . \tag{A.2}$$

The main result of this appendix is:

Proposition A.1.1. *No solution to recurrence* (A.1) *grows faster than the unique solution satisfying initial condition* (A.2).

Here, comparing the rate of growth of two solutions is achieved by comparing their orders $\Theta(\bullet)$.

A.2 The general solution

The general solution to the recurrence (A.1) is a linear combination of the solutions provided by the following result:

Lemma A.2.1. *Let $P(x) = 0$ denote the auxiliary equation of recurrence* (A.1) *and suppose $\lambda \neq 0$ is a root of P with multiplicity d. Then $x_r = \lambda^r r^k$ is a solution of* (A.1) *for each $0 \leq k \leq d-1$.*

Proof. Write

$$P(x) = \sum_{i=0}^{n} p_i x^i \, .$$

Define an operator $T\colon \mathbb{C}[X] \to \mathbb{C}[X]$ by

$$[T(F)](x) = x\frac{\mathrm{d}}{\mathrm{d}x}F(x) \, .$$

Suppose that $\mu \neq 0$ is a root of F with multiplicity $s \geq 2$. Then μ is a root of $T(F)$ with multiplicity $s-1$. Define $Q_m = T^m(P)$ for each $m \in \mathbb{N}$. By hypothesis, λ is a root of P with multiplicity d. Hence, λ is a root of Q_m for all $m \in \{0, \ldots, d-1\}$. Fix $k \in \{0, \ldots, d-1\}$. Then

$$\begin{aligned}
\sum_{i=0}^{n} p_i \lambda^{r+i}(r+i)^k &= \lambda^r \sum_{i=0}^{n} p_i \lambda^i (r+i)^k \\
&= \lambda^r \sum_{i=0}^{n} p_i \lambda^i \left[\sum_{j=0}^{k} \binom{k}{j} r^j i^{k-j} \right] \\
&= \lambda^r \sum_{j=0}^{k} \binom{k}{j} r^j \sum_{i=0}^{n} p_i \lambda^i i^{k-j} \\
&= \lambda^r \sum_{j=0}^{k} \binom{k}{j} r^j Q_{k-j}(\lambda) \\
&= 0 \, .
\end{aligned}$$

Hence $x_r = \lambda^r r^k$ is a solution of the recurrence $\sum_{i=0}^{n} p_i x_{t+i} = 0$ as required. □

Notice that every root of P must be non-zero since $p_0 \neq 0$. It is clear that the set of solutions detailed in Lemma A.2.1 as λ varies over all roots of P is a basis for the space of solutions to recurrence (A.1).

The lemma also addresses the order of each solution: if the root λ has modulus greater than 1 then the growth is exponential; if it has modulus less than 1 then the corresponding solutions are exponentially decaying; if λ has modulus precisely 1 and has multiplicity n then for every $0 \leq k \leq n-1$ there is a polynomial solution of order k.

A.3 The solution to the initial value problem

Let $P(x) = 0$ be the auxiliary equation of recurrence (A.1) and write the roots of P in ascending order of magnitude and multiplicity $\lambda_1, \ldots, \lambda_s$. In particular, if λ_s has multiplicity m as a root of P, then no solution to recurrence (A.1) grows faster than the solution $x_r = \lambda_s^r r^{m-1}$. Let $x^{(1)}, \ldots, x^{(n)}$ denote the solutions given by Lemma A.2.1, with $x_r^{(n)} = \lambda_s^r r^{m-1}$.

Proposition A.3.1. *No solution to recurrence* (A.1) *grows faster than the unique solution satisfying initial condition* (A.2).

Proof. Let y denote the unique solution to recurrence (A.1) with initial condition (A.2). The solution y has a unique expression as a linear combination of the solutions $x^{(1)}, \ldots, x^{(n)}$ and the growth of y will be equal to the growth of the fastest of the solutions with non-zero coefficient in this expression. It will be shown that the solution $x^{(n)}$ has non-zero coefficient in the expression for y.

Define the matrix $A \in \mathrm{End}(\mathbb{C}^n)$ by

$$A_{ij} = x_{i-n}^{(j)} \, .$$

Notice that the columns of A must be linearly independent because they uniquely define the initial conditions for the corresponding solution. Now, finding the expression for y as a linear combination of the $x^{(i)}$ is equivalent to solving the equation

$$Av = [0, \ldots, 0, 1]^T \, .$$

Consider therefore the augmented matrix

$$\left[\begin{array}{ccc|c} A_{1,1} & \cdots & A_{1,n} & 0 \\ \vdots & & \vdots & \vdots \\ A_{n-1,1} & \cdots & A_{n-1,n} & 0 \\ A_{n,1} & \cdots & A_{n,n} & 1 \end{array}\right] .$$

The aim is to use row operations to reduce this augmented matrix to the form

$$\left[\begin{array}{ccccc|c} 1 & * & \cdots & * & * & 0 \\ 0 & 1 & \cdots & * & * & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * & 0 \\ 0 & 0 & \cdots & 0 & 1 & c \end{array}\right].$$

with $c \neq 0$. Notice that this suffices to prove the theorem.

The first step is to realise that $x^{(1)}, \ldots, x^{(n-1)}$ are a linearly independent set of solutions to the recurrence with auxiliary equation $(x - \lambda_s)^{-1}P(x) = 0$. Thus row operations can be used to reduce the augmented matrix under consideration to one of the form

$$\left[\begin{array}{ccccc|c} 1 & 0 & \cdots & 0 & * & 0 \\ 0 & 1 & \cdots & 0 & * & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * & 0 \\ * & * & \cdots & * & * & 1 \end{array}\right].$$

This can be reduced further to

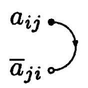
$$\left[\begin{array}{ccccc|c} 1 & 0 & \cdots & 0 & * & 0 \\ 0 & 1 & \cdots & 0 & * & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * & 0 \\ 0 & 0 & \cdots & 0 & q & 1 \end{array}\right].$$

Now $q \neq 0$ since otherwise the rows of A (and hence the columns) would not be linearly independent. This proves the result. $\square$

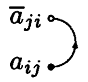
Appendix B

Constructing graph representations of $\underline{\mathbf{Fus}}_{\mathfrak{sl}_3}$

Let Q be a path-connected quiver with Perron-Frobenius eigenvalue $[3]_q$ and corresponding Perron-Frobenius eigenvector x. Consider the path algebra $\mathbb{C}\overline{Q}$ of the doubled quiver $\overline{Q}$. Denote by a_{ij} the arrow $a\colon i \to j$ and denote the dual arrow by $\overline{a}_{ji}$. By analogy with the Temperley-Lieb case, define the annihilation operators as follows:



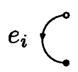
$$= \quad x_j e_i \ ,$$



$$= \quad x_i e_j \ ,$$

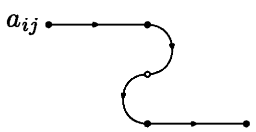
with all other pairs mapping to zero. Define the creation operators by

$$= \quad \frac{1}{x_i} \sum_{c_{ip}} c_{ip} \overline{c}_{pi} \ ,$$

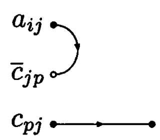


$$= \quad \frac{1}{x_i} \sum_{d_{pi}} \overline{d}_{ip} d_{pi} \ .$$

Then the snake relations are satisfied as the following calculation demonstrates:

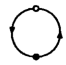
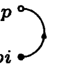


$$= \quad \frac{1}{x_j} \sum_{c_{pj}}$$

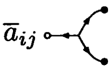


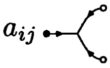
$$= \quad a_{ij} \ .$$

The other snake relations are shown analogously. Furthermore, the following relation holds:

e_i  $= \frac{1}{x_i}\sum_{c_{pi}}$ $\overline{c}_{ip}$ c_{pi}  $= \frac{1}{x_i}\sum_{c_{pi}} x_p e_i = [3]_q e_i\,.$

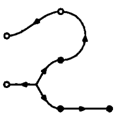
Suppose further that x is a Perron-Frobenius eigenvector for the dual quiver $Q^\vee$. Then the colour-reversed identity holds with an analogous argument to above. Define the remaining two generators by

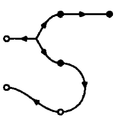
$\overline{a}_{ij}$  $= \sum_{\substack{\Delta_s\\(\overline{a},\overline{b},\overline{c})}} W(\overline{abc})\, c_{ik} b_{kj}\,,$

a_{ij}  $= \sum_{\substack{\Delta_s\\(a,b,c)}} W(abc)\, \overline{c}_{ik}\overline{b}_{kj}\,,$

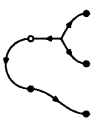
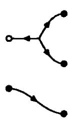
where $W(abc)$ and $W(\overline{abc})$ are constants. Conditions on the weights W are investigated, in order that the operators above extend to a graph representation.

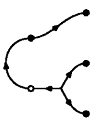
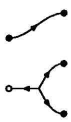
The following calculation shows that the constant $W(\overline{abc})$ must be invariant under cyclic permutation:

$\overline{a}_{ij}$ $\overline{b}_{jk}$  $= \sum_{\substack{\Delta_s\\(\overline{b},\overline{c},\overline{d})}} W(\overline{bcd})$ $\overline{a}_{ij}$ d_{jl} c_{lk}  $= \sum_{\substack{\Delta_s\\(\overline{b},\overline{c},\overline{a})}} W(\overline{bca}) x_j\, c_{ik}\,,$

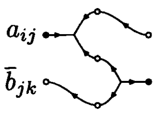
$\overline{a}_{ij}$ $\overline{b}_{jk}$  $= \sum_{\substack{\Delta_s\\(\overline{a},\overline{f},\overline{c})}} W(\overline{afc})$ c_{il} f_{lj} $\overline{b}_{jk}$  $= \sum_{\substack{\Delta_s\\(\overline{a},\overline{b},\overline{c})}} W(\overline{abc}) x_j\, c_{ik}\,.$

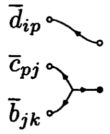
This condition also ensures that the superpotential is well-defined:

e_i  $= \frac{1}{x_i}\sum_{c_{ki}}$ $\overline{c}_{ik}$ c_{ki}  $= \frac{1}{x_i}\sum_{c_{ki}} \sum_{\substack{\Delta_s\\(\overline{c},\overline{b},\overline{a})}} W(\overline{cba})\, a_{ij} b_{jk} c_{ki}\,,$

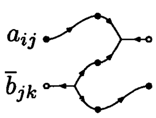
e_i  $= \frac{1}{x_i}\sum_{a_{ij}}$ a_{ij} $\overline{a}_{ji}$  $= \frac{1}{x_i}\sum_{a_{ij}} \sum_{\substack{\Delta_s\\(\overline{a},\overline{c},\overline{b})}} W(\overline{acb})\, a_{ij} b_{jk} c_{ki}\,.$

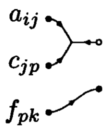
Furthermore, the following two operators coincide, as required:



$= \sum\limits_{\substack{\Delta s\\(a,c,d)}} W(acd)$ 

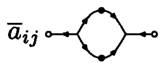
$$= \sum_{\substack{\Delta s\\(a,c,d)}} \sum_{\substack{\Delta s\\(\overline{c},\overline{b},\overline{f})}} W(acd)\, W(\overline{cbf})\, x_j\, \overline{d}_{ip} f_{pk} \,,$$

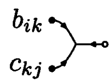


$= \sum\limits_{\substack{\Delta s\\(\overline{b},\overline{f},\overline{c})}} W(\overline{bfc})$ 

$$= \sum_{\substack{\Delta s\\(\overline{b},\overline{f},\overline{c})}} \sum_{\substack{\Delta s\\(a,c,d)}} W(\overline{bfc})\, W(acd)\, x_j\, \overline{d}_{ip} f_{pk} \,.$$

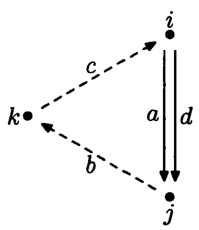
Equality for the colour-reversed operators is shown analogously. The remaining two calculations yield combinatoric equations for the weights:

$[2]_q\, \overline{a}_{ij} =$ 

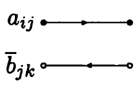
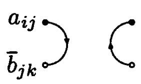
$= \sum\limits_{\substack{\Delta s\\(\overline{a},\overline{c},\overline{b})}} W(\overline{acb})$ 

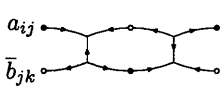
$$= \sum_{\substack{\Delta s\\(\overline{a},\overline{c},\overline{b})}} \sum_{\substack{\Delta s\\(b,c,d)}} W(\overline{acb})\, W(bcd)\, x_k\, \overline{d}_{ij} \,.$$

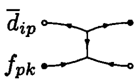
The sum is over the following configuration of triangles:



The colour-reversed identity is derived analogously. The final equation results from the following calculation:

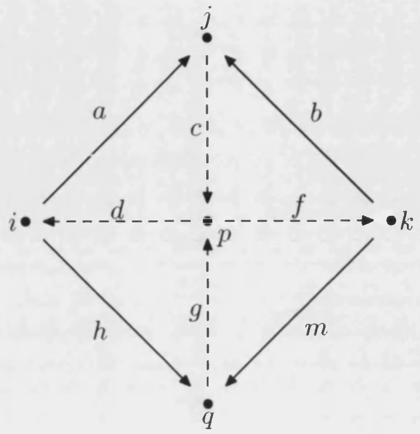
$a_{ij}\overline{b}_{jk} + \frac{x_j}{x_i} \sum\limits_{u_{it}} e_i u_{it} \overline{u}_{ti} e_k =$  $+$ 

$=$ 

$= \sum\limits_{\substack{\Delta s\\(a,c,d)}} \sum\limits_{\substack{\Delta s\\(\overline{b},\overline{f},\overline{c})}} W(\overline{bfc})\, W(acd)\, x_j$ 

$$= \sum_{\substack{\Delta s\\(a,c,d)}} \sum_{\substack{\Delta s\\(\overline{b},\overline{f},\overline{c})}} \sum_{\substack{\Delta s\\(\overline{d},\overline{g},\overline{h})}} \sum_{\substack{\Delta s\\(f,m,g)}} W(\overline{bfc})\, W(acd)\, W(\overline{dgh})\, W(fmg)\, x_j x_p\, h_{iq} \overline{m}_{qk} \,.$$

This last equation looks particularly complicated and is easier visualised as a sum over the following configuration of triangles:



The colour-reversed identity is derived analogously. In conclusion, the following result has been shown:

Theorem B.0.2. *Let Q be a path-connected quiver with Perron-Frobenius eigenvalue $[3]_q$ and Perron-Frobenius eigenvector x and suppose that x is also a Perron-Frobenius eigenvector for the dual quiver $Q^\vee$. A sufficient condition for the existence of a graph representation supported by Q is the existence of weights $W(abc)$ and $W(\overline{cba})$ for each triangle (a,b,c) in the quiver Q satisfying the following equations:*

- $W(abc) = W(cab)$,
  $W(\overline{cba}) = W(\overline{acb})$,

- $\sum\limits_{b,c} W(abc)W(\overline{cba})x_{h(b)} = [2]_q$,
  $\sum\limits_{b,c} W(bcd)W(\overline{cba})x_{h(b)} = 0$ *if* $a \neq d$,
  $\sum\limits_{b,c} W(\overline{dcb})W(abc)x_{h(b)} = 0$ *if* $a \neq d$,

- $\sum\limits_{c,d,f,g} W(acd)W(\overline{bfc})W(gfm)W(\overline{dgh})x_{t(c)}x_{h(c)}$

$$= \begin{cases} 1 & \text{if } a = h, b = m \text{ and } a \neq \overline{b}\ , \\ 0 & \text{if } (a \neq h \text{ or } b \neq m) \text{ and } (a \neq \overline{b})\ , \\ 1 + \frac{x_{h(a)}}{x_{t(a)}} & \text{if } h = a = \overline{b} = \overline{m}\ , \\ \frac{x_{h(a)}}{x_{t(a)}} & \text{if } (a \neq h \text{ or } b \neq m) \text{ and } (a = \overline{b})\ , \end{cases}$$

- $\sum\limits_{c,d,f,g} W(\overline{acd})W(bfc)W(\overline{gfm})W(dgh)x_{h(c)}x_{t(c)}$

$$= \begin{cases} 1 & \text{if } a = h, b = m \text{ and } a \neq \overline{b}\ , \\ 0 & \text{if } (a \neq h \text{ or } b \neq m) \text{ and } (a \neq \overline{b})\ , \\ 1 + \frac{x_{t(a)}}{x_{h(a)}} & \text{if } h = a = \overline{b} = \overline{m}\ , \\ \frac{x_{t(a)}}{x_{h(a)}} & \text{if } (a \neq h \text{ or } b \neq m) \text{ and } (a = \overline{b})\ . \end{cases}$$

It appears that these formulae, in their geometric interpretation as configurations of triangles, arise in a collection of slides for a talk given by Ocneanu at MSRI, available (correct on 13/6/2007) at http://www.msri.org/publications/ln/msri/2000/subfactors/ocneanu ([Ocn00]). Ocneanu claims that the Di Francesco-Zuber quivers of Figures 3-4 and 3-5 (with the notable exception of the quiver with a question mark) satisfy the requisite equations. Thus, one is led to believe that there are graph representations of $\underline{\mathrm{Fus}}_{\mathfrak{sl}_3}$ associated to each (but one) of the Di Francesco-Zuber quivers.

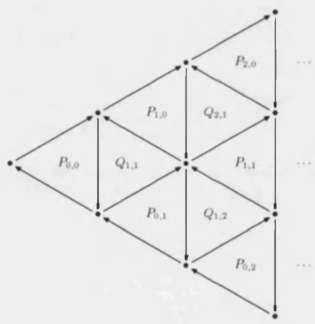
Appendix C

Weight calculations for the $c < 2$ RBCFT quivers

Weights are found for the $\mathcal{A}^{(h)}$ series of $\widehat{\mathfrak{sl}}_3$ rational boundary conformal field theory quivers.

C.1 The $\mathcal{A}^{(h)}$ quivers

Consider the following diagram of the $\mathcal{A}^{(h)}$ quiver, giving labels to each of the triangles within the quiver:



There will be a common abuse of notation by writing:

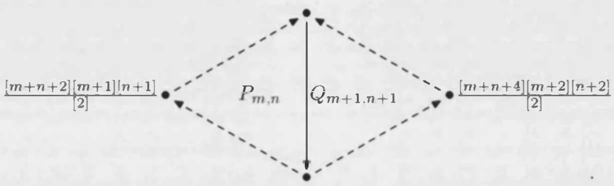
$$P_{m,n} = W(P_{m,n})W(\overline{P_{m,n}}) , \qquad Q_{m,n} = W(Q_{m,n})W(\overline{Q_{m,n}}) .$$

There should be no confusion arising from using the same name for the triangle and for the product of the weights of the triangle and its dual.

Define

$$P_{m,n} = \frac{[2]}{[m+n+3][m+1][n+1]}\ , \qquad Q_{m,n} = \frac{[2]}{[m+n+1][m+1][n+1]}\ .$$

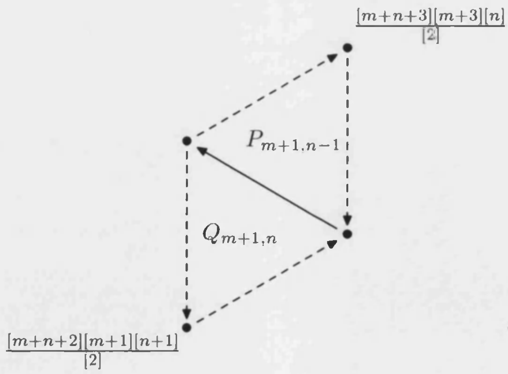
These values for $P_{m,n}$ and $Q_{m,n}$ satisfy all the required equations. Explicitly, consider the following configuration of triangles:



The corresponding weight calculation is

$$\begin{aligned}
&P_{m,n}\frac{[m+n+2][m+1][n+1]}{[2]} \\
&\qquad + Q_{m+1,n+1}\frac{[m+n+4][m+2][n+2]}{[2]} &=& \frac{[2]\cdot[m+n+2][m+1][n+1]}{[m+n+3][m+1][n+1]\cdot[2]} \\
&&& + \frac{[2]\cdot[m+n+4][m+2][n+2]}{[m+n+3][m+2][n+2]\cdot[2]} \\
&&=& \frac{[m+n+2]+[m+n+4]}{[m+n+3]} \\
&&=& [2]\ .
\end{aligned}$$

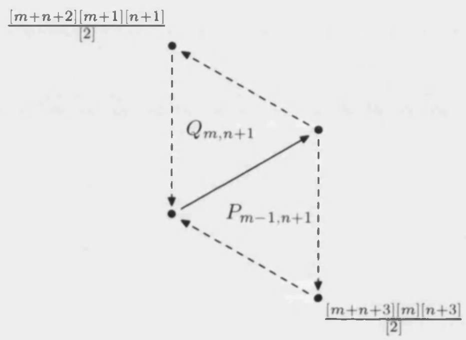
The next configuration of triangles to consider is



The weight calculation is

$$
\begin{aligned}
&Q_{m+1,n}\frac{[m+n+2][m+1][n+1]}{[2]} \\
&\qquad + P_{m+1,n-1}\frac{[m+n+3][m+3][n]}{[2]} &=\;& \frac{[2]\cdot[m+n+2][m+1][n+1]}{[m+n+2][m+2][n+1]\cdot[2]} \\
&&& \quad + \frac{[2]\cdot[m+n+3][m+3][n]}{[m+n+3][m+2][n]\cdot[2]} \\
&&=\;& \frac{[m+1]+[m+3]}{[m+2]} \\
&&=\;& [2]\ .
\end{aligned}
$$

The final configuration of this type is

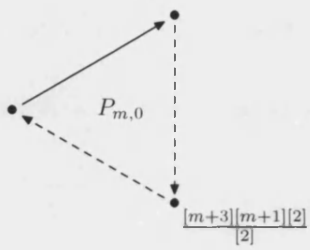


.

The corresponding weight calculation is

$$\begin{aligned}
Q_{m,n+1}\frac{[m+n+2][m+1][n+1]}{[2]} & \\
+ P_{m-1,n+1}\frac{[m+n+3][m][n+3]}{[2]} &= \frac{[2]\cdot[m+n+2][m+1][n+1]}{[m+n+2][m+1][n+2]\cdot[2]} \\
&\quad + \frac{[2]\cdot[m+n+3][m][n+3]}{[m+n+3][m][n+2]\cdot[2]} \\
&= \frac{[n+1]+[n+3]}{[n+2]} \\
&= [2]\ .
\end{aligned}$$

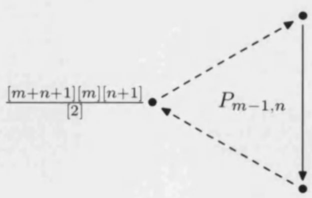
Now check the boundary weights. The boundary triangle



gives rise to the calculation

$$\begin{aligned}
P_{m,0}\frac{[m+3][m+1][2]}{[2]} &= \frac{[2]\cdot[m+3][m+1]}{[m+3][m+1]} \\
&= [2]\ .
\end{aligned}$$

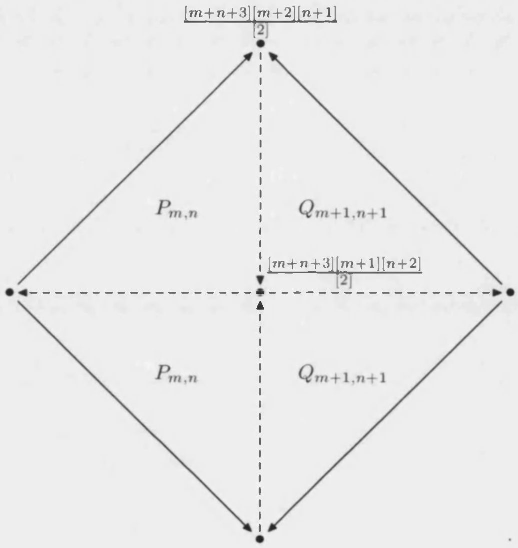
The boundary calculation for $P_{0,m}$ is identical. For the final boundary calculation, suppose that $m+n=h-3$. Then the boundary triangle



yields the calculation

$$
\begin{aligned}
P_{m-1,n}\frac{[m+n+1][m][n+1]}{[2]} &= \frac{[2]\cdot[m+n+1][m][n+1]}{[m+n+2][m][n+1]\cdot[2]} \\
&= \frac{[m+n+1]}{[m+n+2]} \\
&= [2]\ .
\end{aligned}
$$

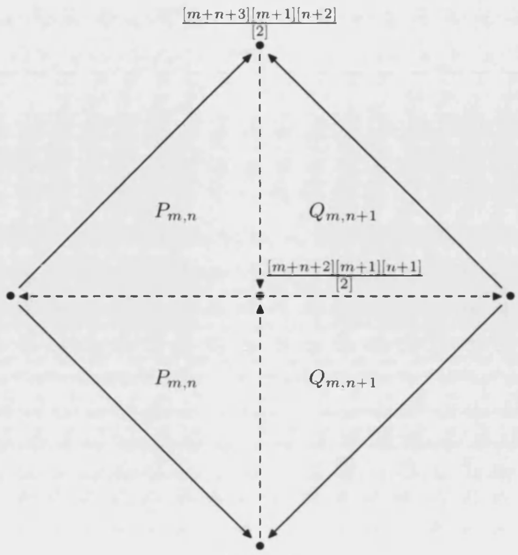
It remains to consider the diagrams associated with the twist operators. Consider the following configuration of triangles:



The corresponding calculation is

$$
\begin{aligned}
P_{m,n}Q_{m+1,n+1} \quad & \frac{[m+n+3][m+2][n+1]}{[2]}\frac{[m+n+3][m+1][n+2]}{[2]} \\
&= \frac{[2]\cdot[2]\cdot[m+n+3][m+2][n+1]\cdot[m+n+3][m+1][n+2]}{[m+n+3][m+1][n+1]\cdot[m+n+3][m+2][n+2]\cdot[2]\cdot[2]} \\
&= 1\ ,
\end{aligned}
$$

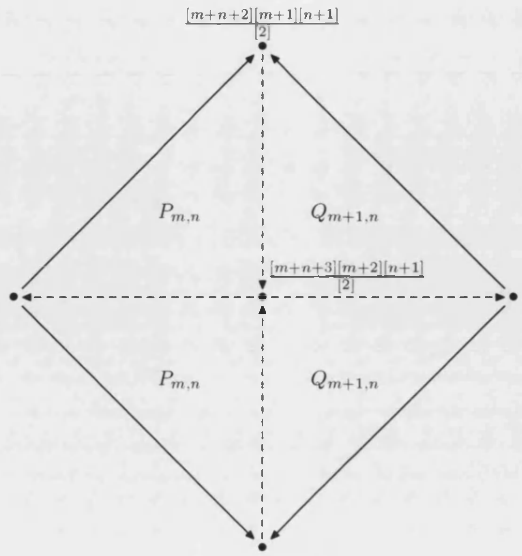
as required. Similarly, the configuration



yields the calculation

$$\begin{aligned} P_{m,n}Q_{m,n+1} \quad & \frac{[m+n+3][m+1][n+2]}{[2]} \frac{[m+n+2][m+1][n+1]}{[2]} \\ &= \frac{[2]\cdot[2]\cdot[m+n+3][m+1][n+2]\cdot[m+n+2][m+1][n+1]}{[m+n+3][m+1][n+1]\cdot[m+n+2][m+1][n+2]\cdot[2]\cdot[2]} \\ &= 1 \end{aligned}$$

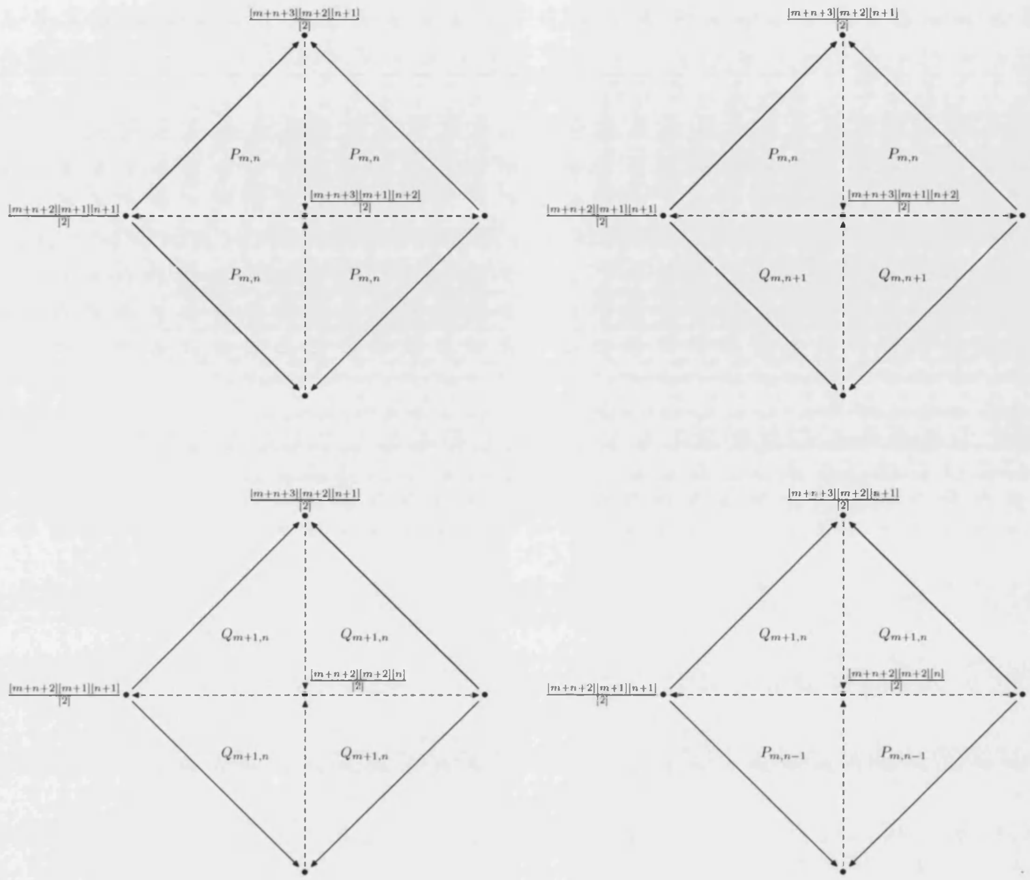
and the configuration



gives rise to the weight calculation

$$
\begin{aligned}
P_{m,n}Q_{m+1,n} & \frac{[m+n+2][m+1][n+1]}{[2]}\frac{[m+n+3][m+2][n+1]}{[2]} \\
&= \frac{[2]\cdot[2]\cdot[m+n+2][m+1][n+1]\cdot[m+n+3][m+2][n+1]}{[m+n+3][m+1][n+1]\cdot[m+n+2][m+2][n+1]\cdot[2]\cdot[2]} \\
&= 1\ .
\end{aligned}
$$

The colour-reversed configurations of this form yield identical equations. The final remaining weight calculations arise from twisting an arrow and its dual. Consider the following configurations of arrows:



These configurations give rise to the following three weight equations:

$$P_{m,n}Q_{m,n+1} \qquad \frac{[m+n+3][m+2][n+1]}{[2]} \frac{[m+n+3][m+1][n+2]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+3][m+2][n+1]\cdot[m+n+3][m+1][n+2]}{[m+n+3][m+1][n+1]\cdot[m+n+2][m+1][n+2]\cdot[2]\cdot[2]}$$

$$= \frac{[m+n+3][m+2]}{[m+n+2][m+1]},$$

$$Q_{m+1,n}P_{m,n-1} \quad \frac{[m+n+3][m+2][n+1]}{[2]} \frac{[m+n+2][m+2][n]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+3][m+2][n+1]\cdot[m+n+2][m+2][n]}{[m+n+2][m+2][n+1]\cdot[m+n+2][m+1][n]\cdot[2]\cdot[2]}$$

$$= \frac{[m+n+3][m+2]}{[m+n+2][m+1]} ,$$

$$(P_{m,n})^2 \quad \frac{[m+n+3][m+2][n+1]}{[2]} \frac{[m+n+3][m+1][n+2]}{[2]}$$

$$+ (Q_{m+1,n})^2 \frac{[m+n+3][m+2][n+1]}{[2]} \frac{[m+n+2][m+2][n]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+3][m+2][n+1]\cdot[m+n+3][m+1][n+2]}{[m+n+3][m+1][n+1]\cdot[m+n+3][m+1][n+1]\cdot[2]\cdot[2]}$$

$$+ \frac{[2]\cdot[2]\cdot[m+n+3][m+2][n+1]\cdot[m+n+2][m+2][n]}{[m+n+2][m+2][n+1]\cdot[m+n+2][m+2][n+1]\cdot[2]\cdot[2]}$$

$$= \frac{[m+2][n+2]}{[m+1][n+1]} + \frac{[m+n+3][n]}{[m+n+2][n+1]} .$$

Now

$$\frac{[m+2][n+1]}{[m+1][n+1]} + \frac{[m+n+3][n]}{[m+n+2][n+1]} = 1 + \frac{[m+n+3][m+2]}{[m+n+2][m+1]}$$

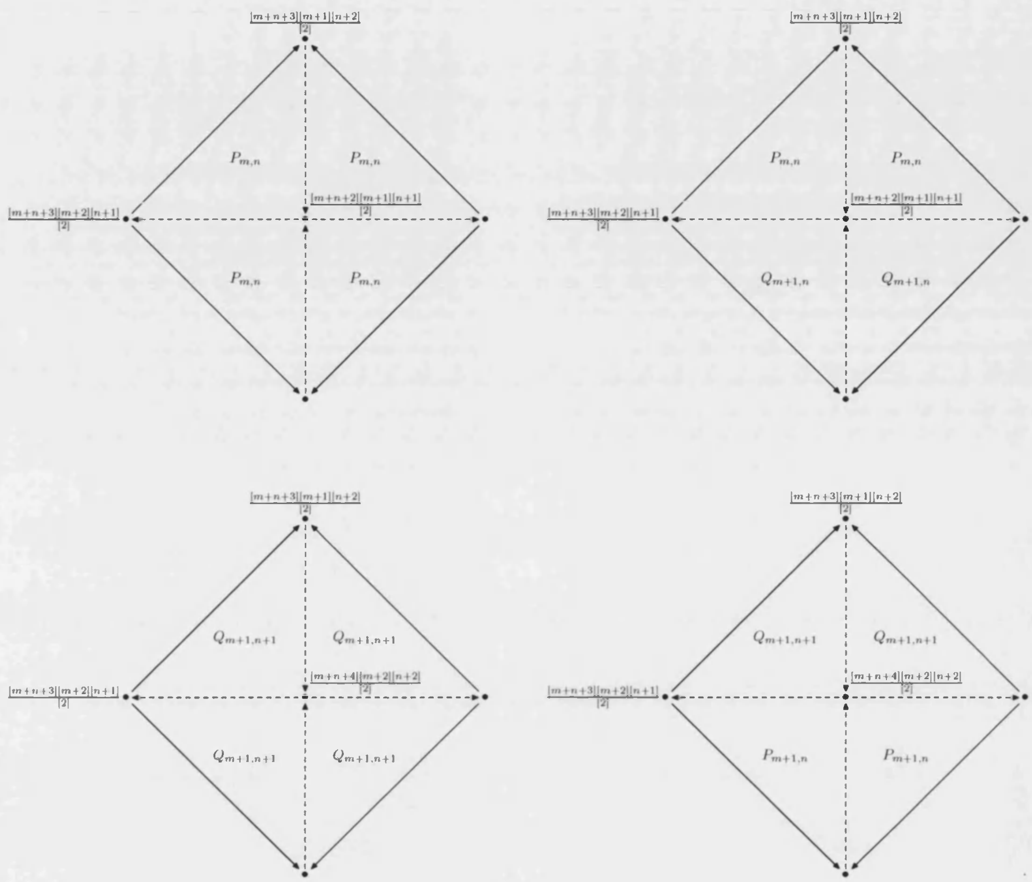
$$\Leftrightarrow ([m+2][n+2] - [m+1][n+1])[m+n+2]$$

$$= ([m+2][n+1] - [m+1][n])[m+n+3] ,$$

which holds due to the formula

$$[r][s] = \sum_{k=1}^{r} [s+r-2k+1] . \tag{C.1}$$

The next set of weight calculations are due to the following configurations of arrows:



The corresponding weight equations are:

$$P_{m,n}Q_{m+1,n} \quad \frac{[m+n+3][m+1][n+2]}{[2]}\frac{[m+n+2][m+1][n+1]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+3][m+1][n+2]\cdot[m+n+2][m+1][n+1]}{[m+n+3][m+1][n+1]\cdot[m+n+2][m+2][n+1]\cdot[2]\cdot[2]}$$

$$= \frac{[m+1][n+2]}{[m+2][n+1]}\ ,$$

$$Q_{m+1,n+1}P_{m+1,n} \quad \frac{[m+n+3][m+1][n+2]}{[2]}\frac{[m+n+4][m+2][n+2]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+3][m+1][n+2]\cdot[m+n+4][m+2][n+2]}{[m+n+3][m+2][n+2]\cdot[m+n+4][m+2][n+1]\cdot[2]\cdot[2]}$$

$$= \frac{[m+1][n+2]}{[m+2][n+1]},$$

$$(P_{m,n})^2 \quad \frac{[m+n+3][m+1][n+2]}{[2]}\frac{[m+n+2][m+1][n+1]}{[2]}$$

$$+ (Q_{m+1,n+1})^2\frac{[m+n+3][m+1][n+2]}{[2]}\frac{[m+n+4][m+2][n+2]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+3][m+1][n+2]\cdot[m+n+2][m+1][n+1]}{[m+n+3][m+1][n+1]\cdot[m+n+3][m+1][n+1]\cdot[2]\cdot[2]}$$

$$+ \frac{[2]\cdot[2]\cdot[m+n+3][m+1][n+2]\cdot[m+n+4][m+2][n+2]}{[m+n+3][m+2][n+2]\cdot[m+n+3][m+2][n+2]\cdot[2]\cdot[2]}$$

$$= \frac{[n+2][m+n+2]}{[n+1][m+n+3]} + \frac{[m+1][m+n+4]}{[m+2][m+n+3]}.$$

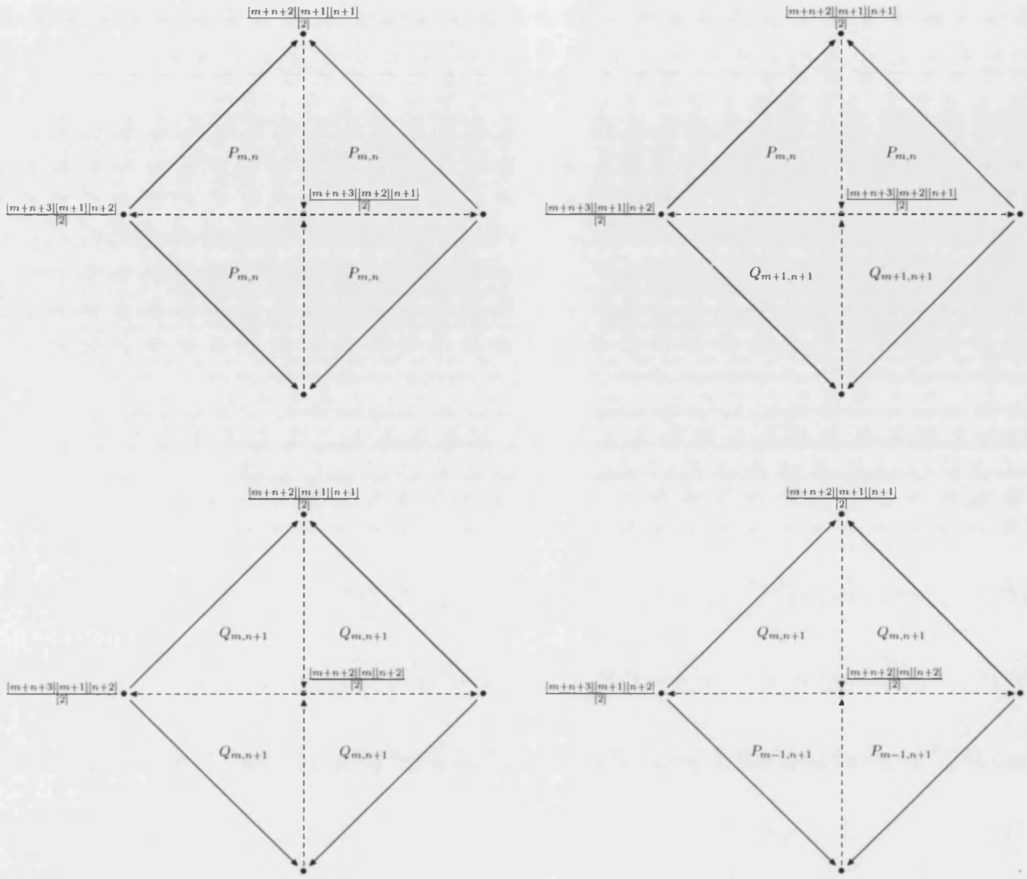
Now

$$\frac{[n+2][m+n+2]}{[n+1][m+n+3]} + \frac{[m+1][m+n+4]}{[m+2][m+n+3]} = 1 + \frac{[m+1][n+2]}{[m+2][n+1]}$$

$$\Leftrightarrow ([m+n+3][m+2] - [m+n+4][m+1])[n+1]$$

$$= ([m+n+2][m+2] - [m+n+3][m+1])[n+2],$$

which can easily be verified using (C.1). The final non-boundary configurations of this type are the following:



The corresponding weight calculations are:

$$P_{m,n}Q_{m+1,n+1} \qquad \frac{[m+n+2][m+1][n+1]}{[2]}\,\frac{[m+n+3][m+2][n+1]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+2][m+1][n+1]\cdot[m+n+3][m+2][n+1]}{[m+n+3][m+1][n+1]\cdot[m+n+3][m+2][n+2]\cdot[2]\cdot[2]}$$

$$= \frac{[m+n+2][n+1]}{[m+n+3][n+2]}\,,$$

$$Q_{m,n+1}P_{m-1,n+1} \quad \frac{[m+n+2][m+1][n+1]}{[2]}\frac{[m+n+2][m][n+2]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+2][m+1][n+1]\cdot[m+n+2][m][n+2]}{[m+n+2][m+1][n+2]\cdot[m+n+3][m][n+2]\cdot[2]\cdot[2]}$$

$$= \frac{[m+n+2][n+1]}{[m+n+3][n+2]}\,,$$

$$(P_{m,n})^2 \quad \frac{[m+n+2][m+1][n+1]}{[2]}\frac{[m+n+3][m+2][n+1]}{[2]}$$

$$+\ (Q_{m,n+1})^2\frac{[m+n+2][m+1][n+1]}{[2]}\frac{[m+n+2][m][n+2]}{[2]}$$

$$= \frac{[2]\cdot[2]\cdot[m+n+2][m+1][n+1]\cdot[m+n+3][m+2][n+1]}{[m+n+3][m+1][n+1]\cdot[m+n+3][m+1][n+1]\cdot[2]\cdot[2]}$$

$$+\frac{[2]\cdot[2]\cdot[m+n+2][m+1][n+1]\cdot[m+n+2][m][n+2]}{[m+n+2][m+1][n+2]\cdot[m+n+2][m+1][n+2]\cdot[2]\cdot[2]}$$

$$= \frac{[m+n+2][m+2]}{[m+n+3][m+1]}+\frac{[m][n+1]}{[m+1][n+2]}\,.$$

Now

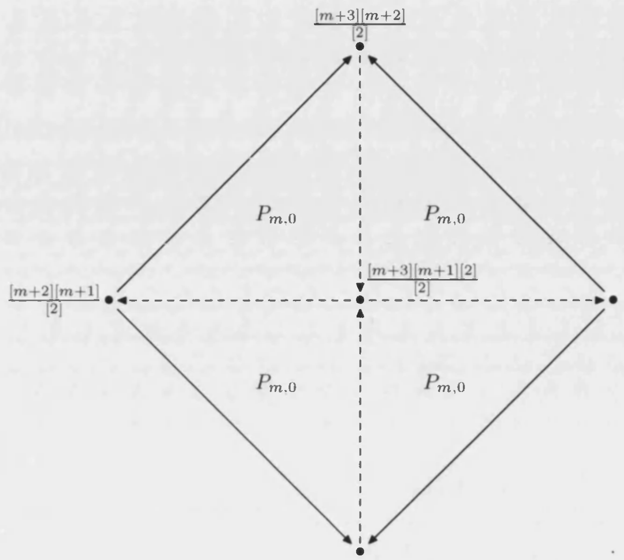
$$\frac{[m+n+2][m+2]}{[m+n+3][m+1]}+\frac{[m][n+1]}{[m+1][n+2]} \quad = \quad 1+\frac{[m+n+2][n+1]}{[m+n+3][n+2]}$$

$$\Leftrightarrow ([m+1][n+2]-[m][n+1])[m+n+3]$$

$$= ([m{+}2][n{+}2]-[m{+}1][n{+}1])[m{+}n{+}2]\,,$$

which can easily be verified using (C.1).

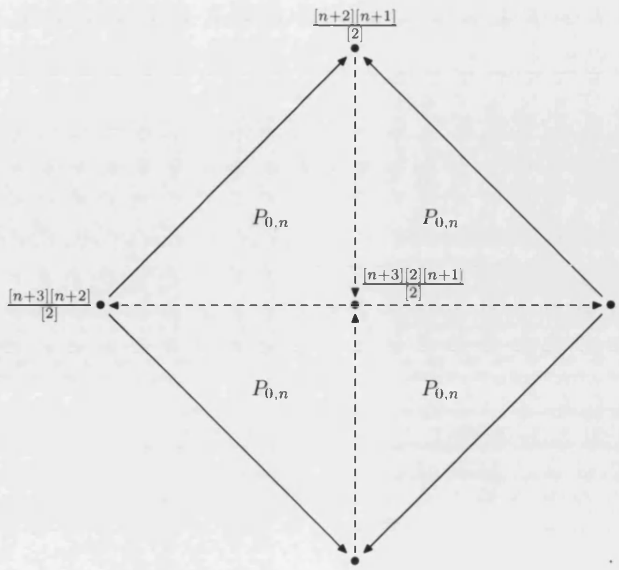
It remains only to check the equations for these configurations on the boundary. The equation at the cutoff is a special case of an equation that has already been considered (fixing h simply forces one of the eigenvector components to be zero in the calculation). For the top boundary, the pertinent configuration of arrows is



The corresponding weight calculation is thus

$$\begin{aligned}
(P_{m,0})^2 \frac{[m+3][m+2]}{[2]} \frac{[m+3][m+1][2]}{[2]} &= \frac{[2] \cdot [2] \cdot [m+3][m+2] \cdot [m+3][m+1][2]}{[m+3][m+1] \cdot [m+3][m+1] \cdot [2] \cdot [2]} \\
&= \frac{[m+2][2]}{[m+1]} \\
&= 1 + \frac{[m+3]}{[m+1]} .
\end{aligned}$$

Finally it remains to check consider the following configuration of arrows:



$$
\begin{aligned}
(P_{0,n})^2 \frac{[n+2][n+1]}{[2]} \frac{[n+3][2][n+1]}{[2]} &= \frac{[2]\cdot[2]\cdot[n+2][n+1]\cdot[n+3][2][n+1]}{[n+3][n+1]\cdot[n+3][n+1]\cdot[2]\cdot[2]} \\
&= \frac{[n+2][2]}{[n+3]} \\
&= 1 + \frac{[n+1]}{[n+3]} \ .
\end{aligned}
$$

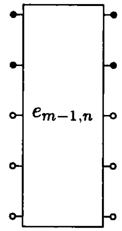
This concludes all the necessary calculations. Hence the weights $P_{m,n}$ and $Q_{m,n}$ define a graph representation. The choice of $W(\overline{P_{m,n}}) \neq 0$ and $W(\overline{Q_{m,n}}) \neq 0$ is arbitrary.

Appendix D

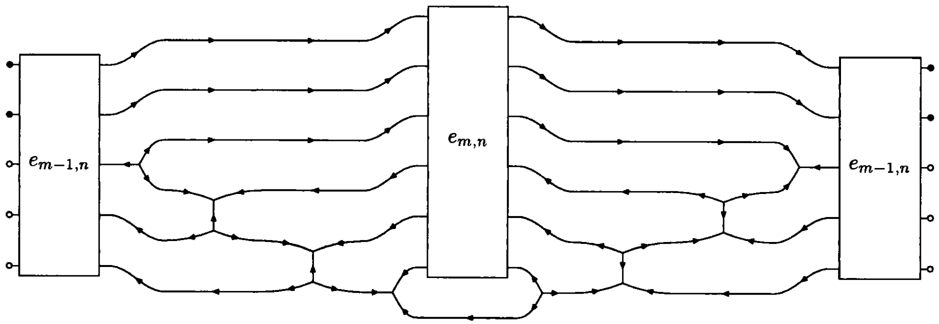
Some lengthy calculations

This chapter contains a number of calculations that need to be broken over several pages and would constitute a barrier to understanding if typeset in the main text. The calculations are all diagrammatic and benefit from being presented in a landscape orientation.

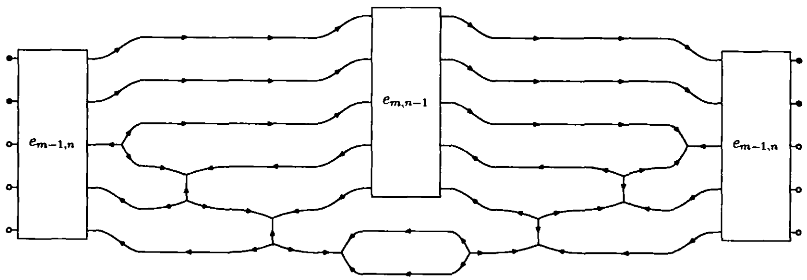
Calculation D.1. *From page 146,*

$$\phi^\circ\phi \quad = \quad \frac{[m+1][m+n+2]}{[m][m+n+1]}$$ 

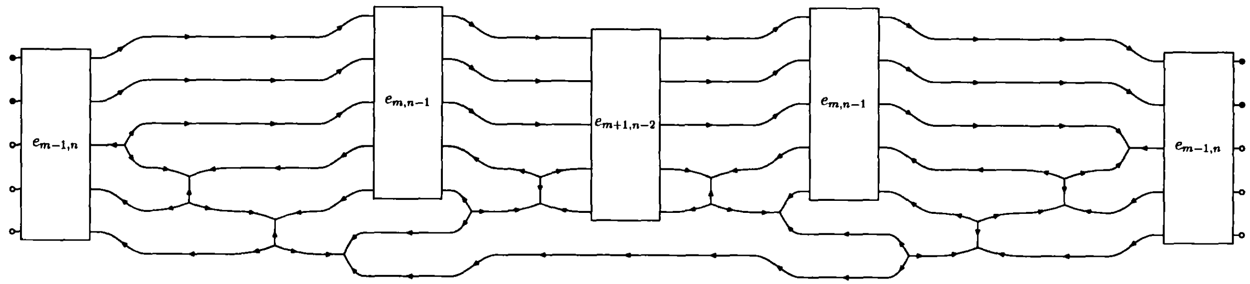
Proof. By definition,

$$\phi^\circ\phi \quad =$$ 

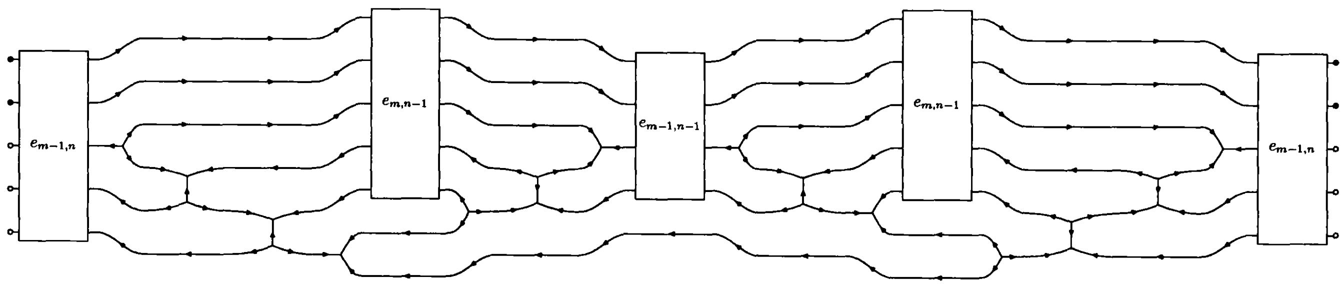
$$=$$

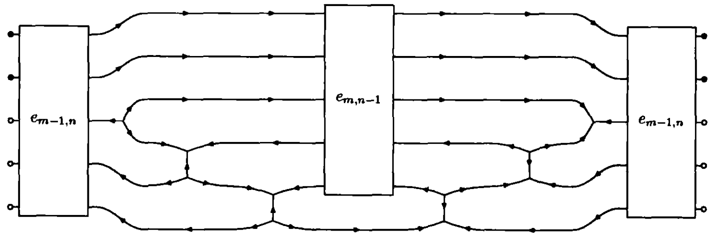


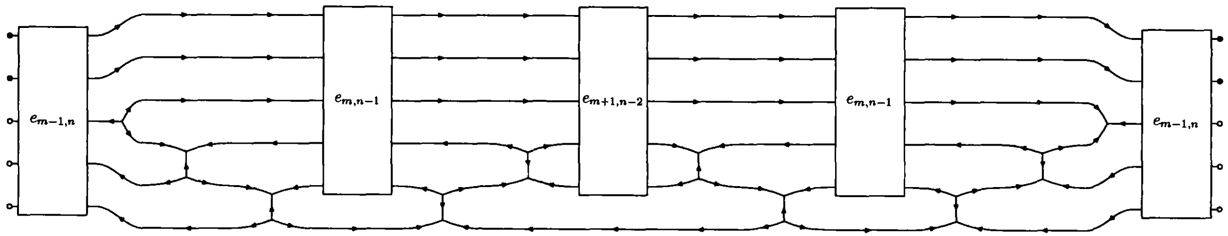
$$-\frac{[n-1]}{[n]}$$

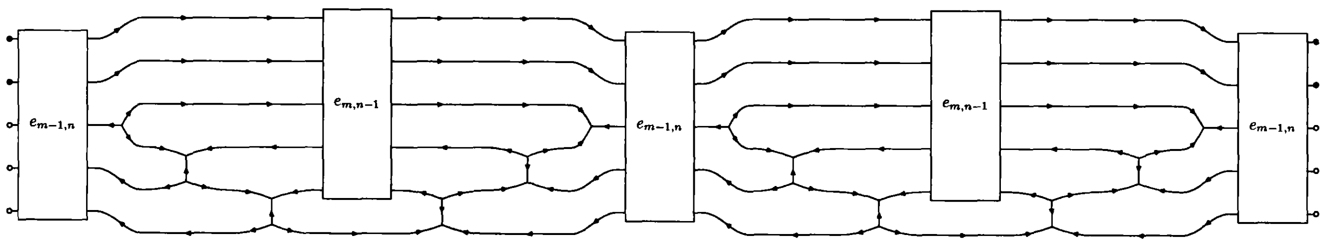


$$-\frac{[m][m+n]}{[m+1][m+n+1]}$$

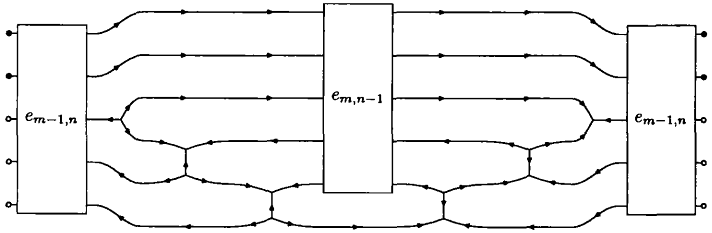


$= \quad [2]$ 

$- \quad \frac{[n-1]}{[n]}$ 

$- \quad \frac{[m][m+n]}{[m+1][m+n+1]}$ 

Simplify these components one at a time:



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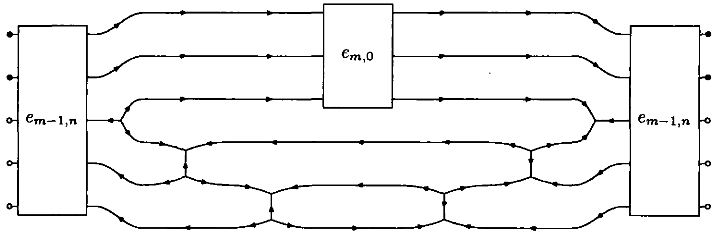
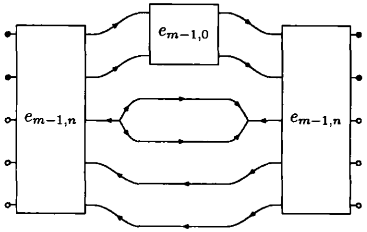
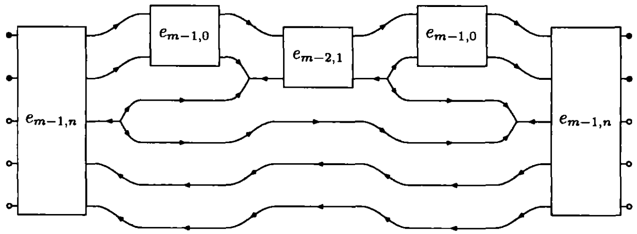


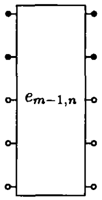
$=$



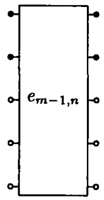
$- \frac{[m-1]}{[m]}$



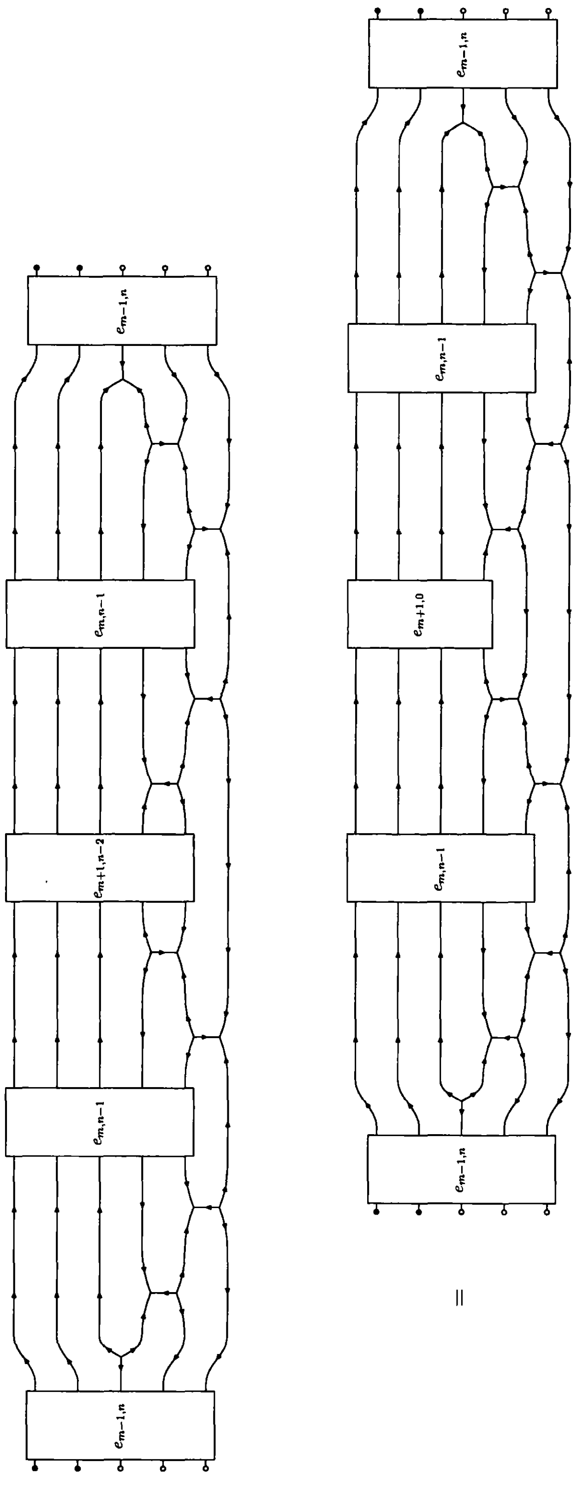
$= \left([2] - \frac{[m-1]}{[m]}\right)$



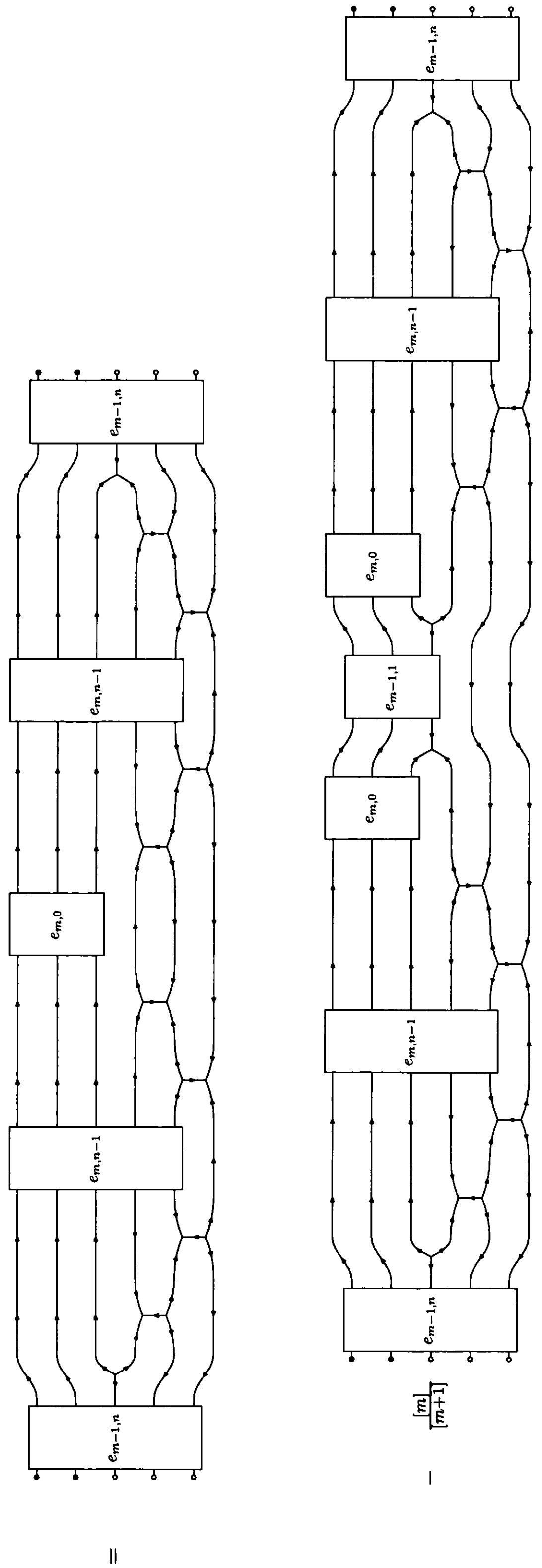
$= \frac{[m+1]}{[m]}$



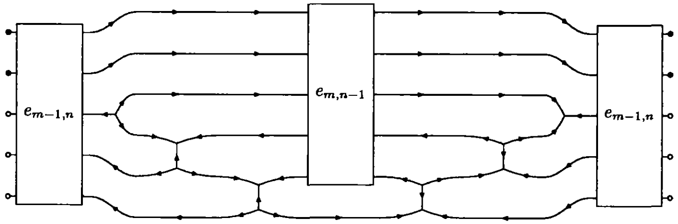
Now the next component:



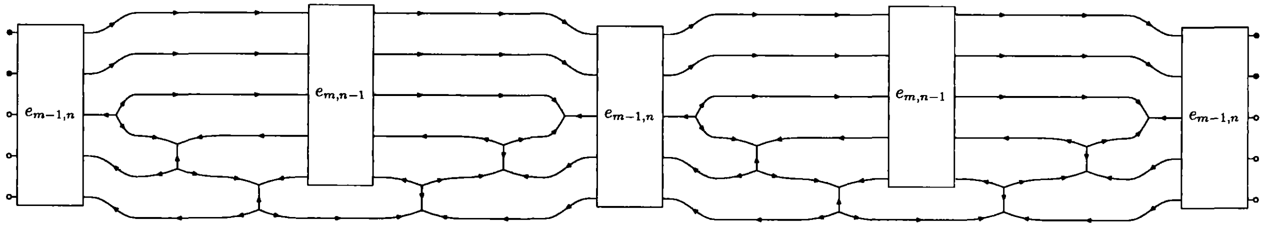
$e_{m-1,n}$
$e_{m,n-1}$
$e_{m+1,n-2}$
$e_{m,n-1}$
$e_{m-1,n}$
=
$e_{m-1,n}$
$e_{m,n-1}$
$e_{m+1,0}$
$e_{m,n-1}$
$e_{m-1,n}$


$e_{m-1,n}$
$e_{m,n-1}$
$e_{m,0}$
$e_{m-1,1}$
$\frac{[m]}{[m+1]}$

$=$

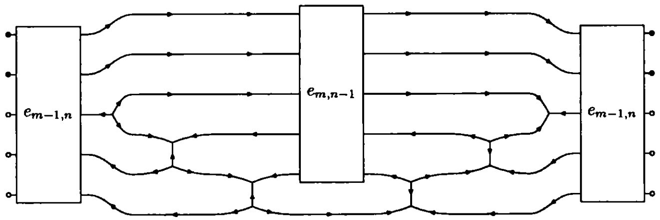


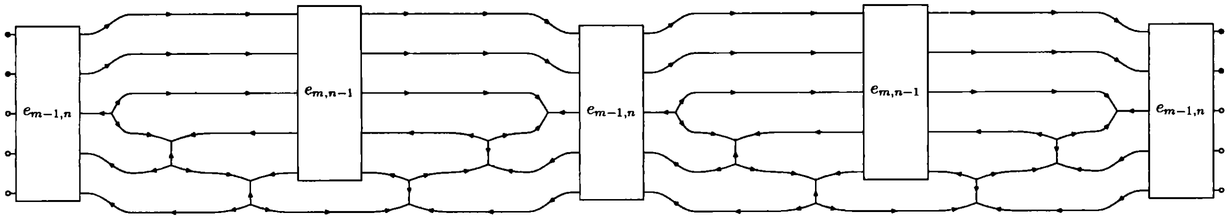
$- \quad \frac{[m]}{[m+1]}$

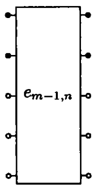


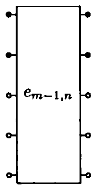
$$= \quad 0\,.$$

Combining these calculations yields

$\phi^\circ\phi \quad = \quad [2]$ 

$- \frac{[m][m+n]}{[m+1][m+n+1]}$ 

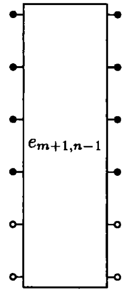
$= \quad \left(\frac{[2][m+1]}{[m]} - \frac{[m+1][m+n]}{[m][m+n+1]}\right)$ 

$= \quad \frac{[m+1][m+n+2]}{[m][m+n+1]}$ 

as required. □

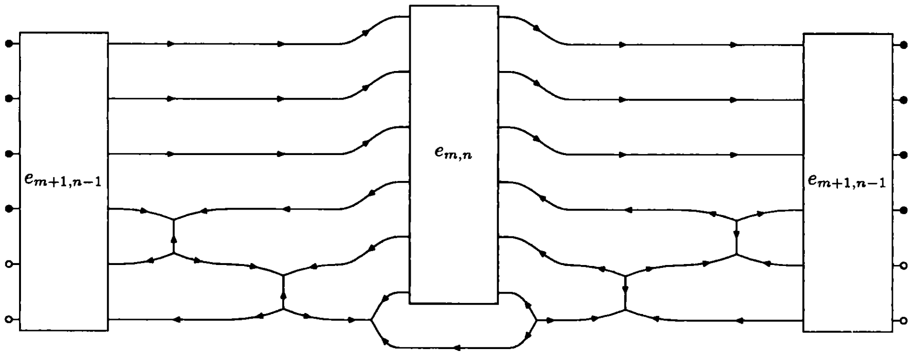
Calculation D.2. *From page 147,*

$$\psi^{\circ}\psi \quad = \quad \frac{[n+1]}{[n]}$$

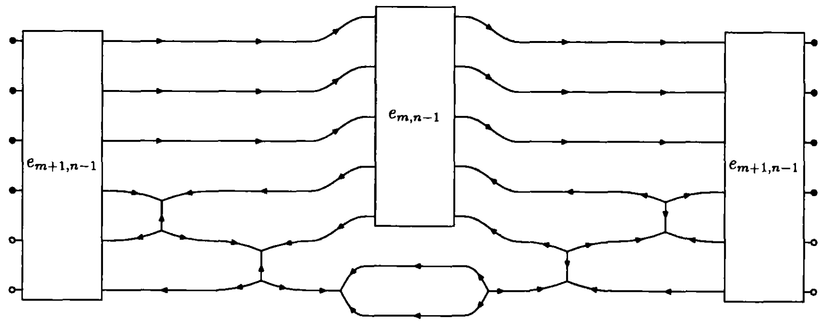


Proof. By definition,

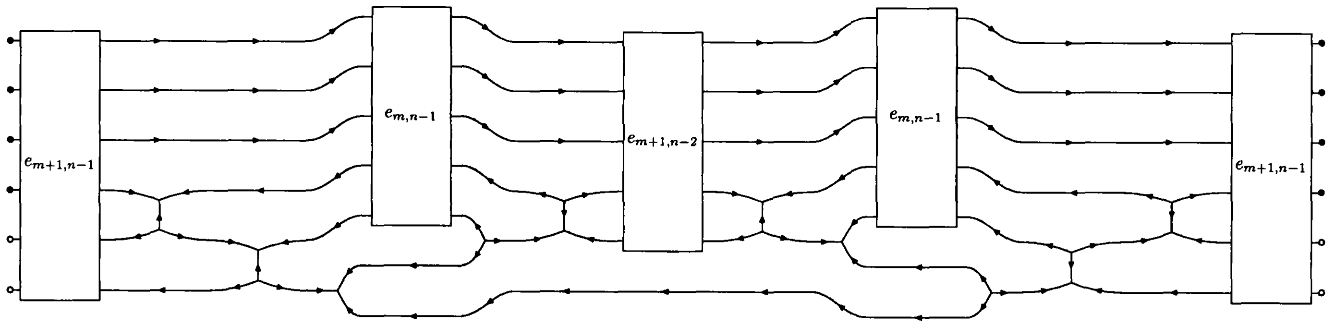
$$\psi^{\circ}\psi \qquad =$$

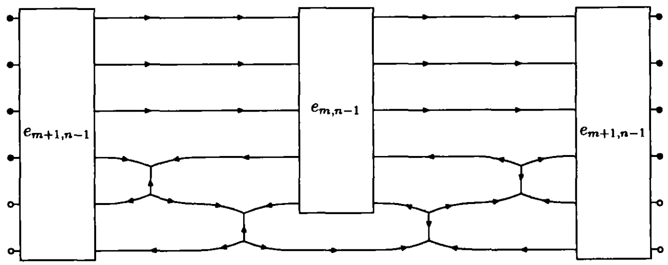


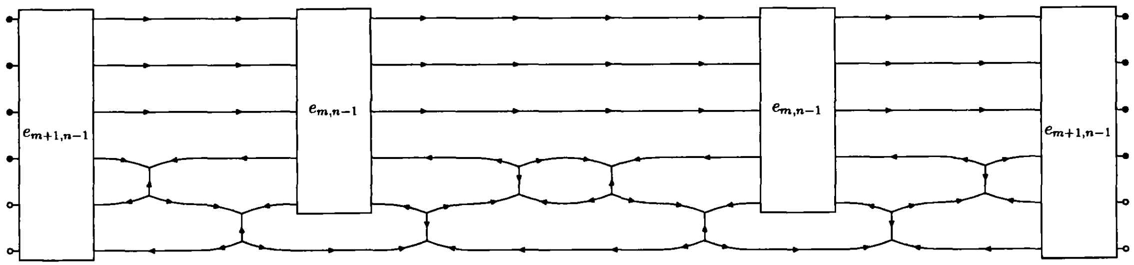
$=$

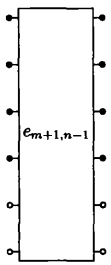


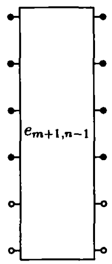
$- \frac{[n-1]}{[n]}$



$=$ $[2]$ 

$-$ $\frac{[n-1]}{[n]}$ 

$=$ $\left([2] - \frac{[n-1]}{[n]}\right)$ 

$=$ $\frac{[n+1]}{[n]}$ 

as required. $\square$

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